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Integral criteria for the existence of positive solutions of first-order linear differential advanced-argument equations

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ABSTRACT

A linear differential equation with advanced-argument $y'(t) - c(t)y(t+\tau) = 0$ is considered where $c\colon [t_0, \infty) \to [0, \infty)$, $t_0 \in \mathbb{R}$ is a bounded and locally Lipschitz continuous function and $\tau > 0$. The well-known explicit integral criterion

$$\int_t^{t+\tau} c(s)\,\mathrm{d}s \le 1/\mathrm{e}, \quad t \in [t_0, \infty)$$

guarantees the existence of a positive solution on $[t_0, \infty)$. The paper derives new integral criteria involving the coefficient c. Their independence of the previous result is discussed as well.

1. Introduction

The paper considers the existence of positive solutions of a linear differential equation with advanced-argument,

$$y'(t) - c(t)y(t+\tau) = 0, \quad t \ge t_0 \tag{1}$$

where $\tau > 0$ and the function $c\colon [t_0, \infty) \to [0, \infty)$ is bounded and locally Lipschitz continuous. Throughout the paper, we assume $t_0 > 0$. This assumption is only technical to guarantee all expressions below being well-defined. Repeating the classical definitions, we say that a function y is solution of (1) on $[t_0, \infty)$ if it is continuously differentiable, and for $t \in [t_0, \infty)$ satisfies (1). A solution y of (1) is called oscillatory if it has arbitrarily large zero points. Otherwise it is called non-oscillatory. A non-oscillatory solution y of (1) is called positive (negative) on $[t_0, \infty)$ if $y(t) > 0$ ($y(t) < 0$) on $[t_0, \infty)$.

In the literature, one can find only few results on the existence of positive solutions of advanced-argument equations. An implicit criterion is given in [1, Theorem 6] where a general theorem on the existence of a

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positive solution to systems of nonlinear functional differential equations of advanced type is proved. The following implicit criterion given in [2, Theorem 1] is its specification to (1).

Theorem 1. *For the existence of a positive solution $y = y(t)$ of (1) on $[t_0, \infty)$, a necessary and sufficient condition is the existence of a continuous function $\lambda\colon [t_0, \infty) \to \mathbb{R}$, satisfying the inequality*

$$\lambda(t) \geq c(t) \exp\left(\int_t^{t+\tau} \lambda(s)\,\mathrm{d}s\right), \quad t \in [t_0, \infty). \tag{2}$$

A proper selection of λ in (2) results in explicit criteria of positivity. For $\lambda(t) := \mathrm{e}\,c(t)$ we derive the following well-known criterion published in [3].

Theorem 2. *For the existence of a positive solution $y = y(t)$ of (1) on $[t_0, \infty)$, the inequality*

$$\int_t^{t+\tau} c(s)\,\mathrm{d}s \leq 1/\mathrm{e}, \quad t \in [t_0, \infty) \tag{3}$$

is sufficient.

We also refer to the paper [4] on oscillation of first-order advanced-argument equations. Unfortunately, there exists no general formula for finding a proper function λ in (2) to get the best possible explicit criteria. Recently, some recommendations resulting in a new class of integral criteria, have been suggested in [2,5]. In [6, Chapter 5], linear advanced-argument differential equations are also considered, in particular, criteria (2) and (3) can be deduced from the results obtained. A criterion for the existence of positive solutions of (1) different from criterion (2) is given in [7, p. 21]. Similar problems on the existence of positive and oscillating solutions of delayed functional differential equations are treated in many papers and books. We refer, e.g., to [6–10], and to the references therein.

The purpose of the paper is to derive new explicit integral criteria for the existence of a positive solution of Eq. (1) and compare them with criterion (3).

2. Positivity criteria

To get new positivity criteria we will employ implicit criterion (2) with $\lambda(t) := \mathrm{e}\,c(t)\exp\omega(t)$ where $\omega\colon [t_0, \infty) \to \mathbb{R}$. Then, an auxiliary lemma holds.

Lemma 1. *Let $\omega\colon [t_0, \infty) \to \mathbb{R}$ be a continuous function such that*

$$\int_t^{t+\tau} c(s)\mathrm{e}^{\omega(s)} ds \leq \frac{1}{\mathrm{e}}\left(1 + \omega(t)\right) \tag{4}$$

for $t \in [t_0, \infty)$. Then, there exists a positive solution $y = y(t)$ of (1) on $[t_0, \infty)$.

We will also need the following auxiliary result, which is a consequence of Lemma 1.

Theorem 3. *Let $n \geq 2$, $\omega\colon [t_0, \infty) \to \mathbb{R}$ be a nonincreasing continuous function and let the inequality*

$$\begin{aligned} &\mathrm{e}^{\omega(t)} \int_t^{t+(1/n)\tau} c(s)ds + \mathrm{e}^{\omega(t+(1/n)\tau)} \int_{t+(1/n)\tau}^{t+(2/n)\tau} c(s)ds + \cdots \\ &+ \mathrm{e}^{\omega(t+((n-1)/n)\tau)} \int_{t+((n-1)/n)\tau}^{t+\tau} c(s)ds \leq \frac{1}{\mathrm{e}}\left(1 + \omega(t)\right) \end{aligned} \tag{5}$$

hold for $t \in [t_0, \infty)$. Then, there exists a positive solution $y = y(t)$ of (1) on $[t_0, \infty)$.