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Hyers–Ulam stability of first-order homogeneous linear differential equations with a real-valued coefficient

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Dedicated to Professor Jitsuro Sugie on the occasion of his 60th birthday

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ABSTRACT

This paper is concerned with the Hyers–Ulam stability of the first-order linear differential equation $x' - ax = 0$, where a is a non-zero real number. The main purpose is to find an explicit solution $x(t)$ of $x' - ax = 0$ satisfying $|\phi(t) - x(t)| \leq \varepsilon/|a|$ for all $t \in \mathbb{R}$ under the assumption that a differentiable function $\phi(t)$ satisfies $|\phi'(t) - a\phi(t)| \leq \varepsilon$ for all $t \in \mathbb{R}$. In addition, the precise behavior of the solutions of $x' - ax = 0$ near the function $\phi(t)$ is clarified on the semi-infinite interval. Finally, some applications to nonhomogeneous linear differential equations are included to illustrate the main result.

1. Introduction

We consider the first-order homogeneous linear differential equation

$$x' - ax = 0, \quad t \in I, \tag{1}$$

where I is a nonempty open interval of $\mathbb{R}$; a is a non-zero real number. We call that Eq. (1) has the "*Hyers–Ulam stability*" on I if there exists a constant $K > 0$ with the following property: Let $\varepsilon > 0$ be a given arbitrary constant. If a differentiable function $\phi : I \to \mathbb{R}$ satisfies $|\phi'(t) - a\phi(t)| \leq \varepsilon$ for all $t \in I$, then there exists a solution $x : I \to \mathbb{R}$ of Eq. (1) such that $|\phi(t) - x(t)| \leq K\varepsilon$ for all $t \in I$. We call such K a "*HUS constant*" for Eq. (1) on I. It is easy to check that if $a = 0$ then Eq. (1) does not have the Hyers–Ulam stability on $\mathbb{R}$. From this reason, we consider only the case that $a \neq 0$.

In 1998, Alsina and Ger [1] studied the Hyers–Ulam stability of the fundamental linear differential equation $x' - x = 0$. They proved that the linear differential equation $x' - x = 0$ has the Hyers–Ulam stability with a HUS constant 3 on I. After that, many researchers have studied the Hyers–Ulam stability of the various linear differential equations (see [2–16]).

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In 2003, Miura, Miyajima and Takahasi [11, Corollary 2.5] gave the following sharp result. The original result can be applied to the Banach space-valued differential equations.

Theorem A. *Eq. (1) has the Hyers–Ulam stability with a HUS constant $1/|a|$ on $\mathbb{R}$. Here, $1/|a|$ is the minimum of HUS constants for Eq. (1) on $\mathbb{R}$.*

Moreover, using one of the results presented by Jung [7], Miura, Miyajima and Takahasi [11], Takahasi, Miura and Miyajima [14], we see that the solution $x(t)$ of (1) satisfying $|\phi(t) - x(t)| \le \varepsilon/|a|$ for all $t \in \mathbb{R}$ is the only one (unique). An important question now arises. Can we find an explicit solution corresponding to the above solution $x(t)$ of (1)? The purpose of this paper is to give the answer to this question. In addition, we will investigate the precise behavior of the solutions of (1) near the function $\phi(t)$, under the assumption that $\sup I$ or $\inf I$ exists. The obtained result is as follows.

Theorem 1. *Let $\varepsilon > 0$ be a given arbitrary constant. Suppose that a differentiable function $\phi : I \to \mathbb{R}$ satisfies $|\phi'(t) - a\phi(t)| \le \varepsilon$ for all $t \in I$. Then one of the following holds:*

(i) *if $a > 0$ and $\sup I$ exists, then $\lim_{t\to\tau-0} \phi(t)$ exists where $\tau = \sup I$, and any solution $x(t)$ of (1) with $|\lim_{t\to\tau-0} \phi(t) - x(\tau)| < \varepsilon/a$ satisfies that $|\phi(t) - x(t)| < \varepsilon/a$ for all $t \in I$;*
(ii) *if $a > 0$ and $\sup I$ does not exist, then $\lim_{t\to\infty} \phi(t)e^{-at}$ exists, and there exists exactly one solution $x(t) = (\lim_{t\to\infty} \phi(t)e^{-at})e^{at}$ of (1) such that $|\phi(t) - x(t)| \le \varepsilon/a$ for all $t \in I$;*
(iii) *if $a < 0$ and $\inf I$ exists, then $\lim_{t\to\sigma+0} \phi(t)$ exists where $\sigma = \inf I$, and any solution $x(t)$ of (1) with $|\lim_{t\to\sigma+0} \phi(t) - x(\sigma)| < \varepsilon/|a|$ satisfies that $|\phi(t) - x(t)| < \varepsilon/|a|$ for all $t \in I$;*
(iv) *if $a < 0$ and $\inf I$ does not exist, then $\lim_{t\to-\infty} \phi(t)e^{-at}$ exists, and there exists exactly one solution $x(t) = (\lim_{t\to-\infty} \phi(t)e^{-at})e^{at}$ of (1) such that $|\phi(t) - x(t)| \le \varepsilon/|a|$ for all $t \in I$.*

From Theorem 1, we can establish the following result.

Corollary 2. *Eq. (1) has the Hyers–Ulam stability with a HUS constant $1/|a|$ on I.*

Remark 1. In the special case that $a = 1$, a HUS constant for Eq. (1) on I is one from Corollary 2. That is, we can conclude that our theorem is an improvement of the result of Alsina and Ger [1].

In the case that $I = \mathbb{R}$, we can state the following result from the assertions (ii) and (iv) in Theorem 1.

Corollary 3. *Eq. (1) has the Hyers–Ulam stability with a HUS constant $1/|a|$ on $\mathbb{R}$. Furthermore, the solution $x(t)$ of (1) satisfying $|\phi(t) - x(t)| \le \varepsilon/|a|$ for all $t \in \mathbb{R}$ is the only one, which written as $x(t) = (\lim_{t\to\infty} \phi(t)e^{-at})e^{at}$ if $a > 0$ (resp., $x(t) = (\lim_{t\to-\infty} \phi(t)e^{-at})e^{at}$ if $a < 0$).*

Remark 2. Let $\varepsilon > 0$ be a given arbitrary constant. We consider the nonhomogeneous differential equation

$$x' - ax = -\varepsilon$$

on $\mathbb{R}$, where a is a non-zero real number. We can easily see that the function $\phi(t) = \varepsilon/a + ce^{at}$ for $t \in \mathbb{R}$ is the general solution of this nonhomogeneous differential equation, where c is an arbitrary constant. Since ce^{at} is a solution of (1), $|\phi(t) - x(t)| = \varepsilon/|a|$ holds for all $t \in \mathbb{R}$. From this fact and the assertion in Corollary 3, we can conclude that $1/|a|$ is the minimum of HUS constants for Eq. (1) on $\mathbb{R}$. Moreover, this example shows that it is not possible to weaken the condition $|\lim_{t\to\tau-0} \phi(t) - x(\tau)| < \varepsilon/a$ in (i) of Theorem 1 to $|\lim_{t\to\tau-0} \phi(t) - x(\tau)| \le \varepsilon/a$, in order to satisfy $|\phi(t) - x(t)| < \varepsilon/a$ for $t \in I$.