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Multiplicity results for the Kirchhoff type equations with critical growth

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ABSTRACT

In this paper, we study the following Kirchhoff type equation with critical growth

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\triangle u = \lambda u + \mu |u|^2 u + |u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a > 0, b \geq 0$ and Ω is a smooth bounded domain in $\mathbb{R}^3$. When the real parameter μ is larger than some positive constant, we investigate the multiplicity of nontrivial solutions for the above problem with parameter λ belonging to a left neighborhood of the Dirichlet eigenvalue of the Laplacian operator $-\triangle$.

1. Introduction

In this paper we investigate the multiplicity of solutions for the following non-local problem:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\triangle u = \lambda u + \mu |u|^2 u + |u|^4 u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a > 0, b \geq 0$ and Ω is a smooth bounded domain in $\mathbb{R}^3$. We remark that $|u|^4 u$ reaches the Sobolev critical exponent since $2^* = 6$ in dimension three. (1.1) is related to the stationary analogue of the following

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equation

$$\begin{cases} u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right) \triangle u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Such problems are viewed as being nonlocal because of the presence of the term $\int_\Omega |\nabla u|^2 \mathrm{d}x$, which implies that the equations in (1.1) and (1.2) are no longer a pointwise identity and are very different from classical elliptic equations. The solvability of the Kirchhoff type equations has been well studied in a general dimension by various authors only after Lions [1] introduced an abstract framework to this problem. For example, see [2–10]. We especially emphasize that, when a nonlinear elliptic boundary value problem has a critical term such as (1.1), a crucial difficulty occurs in proving the existence of solutions of the problem. Such difficulty is caused by the lack of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. In order to overcome this difficulty, Naimen made use of the well-known method of Brezis and Nirenberg [11] to obtain some important existence results, for example, see [12,13]. However, our main aim in this paper is to use a cut-off technique together with an abstract critical point theorem (see [14]) to investigate the multiplicity of nontrivial solutions for (1.1).

Hereafter, we need the following preliminaries. Let $H := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm $\langle u, v\rangle = \int_\Omega \nabla u \nabla v \mathrm{d}x$, $\|u\| := \langle u, v\rangle^{\frac{1}{2}}$, respectively. Let $|\cdot|_s$ be the usual L_s-norm and since Ω is a bounded domain, $H \hookrightarrow L^s(\Omega)$ continuously for $s \in [1, 2^*]$, compactly for $s \in [1, 2^*)$. S denotes the best Sobolev constant: $S := \inf\{\|u\|^2/|u|_{2^*}^2 : u \in H_0^1(\Omega)\setminus\{0\}\}$. For convenience in computation, we use $a\lambda$ and $b\mu$ to denote λ and μ in (1.1), respectively. Therefore, (1.1) can be rewritten as

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right) \triangle u = a\lambda u + b\mu|u|^2u + |u|^4u, & \text{in } \Omega. \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Denote by $0 < \lambda_1 < \lambda_2 \le \cdots$ the Dirichlet eigenvalues of $-\triangle$ on Ω, repeated according to multiplicity. Now we note that $\lambda_1 > S/|\Omega|^{2/3}$. Indeed, let φ_1 be an eigenfunction associated with λ_1. It is known that S is not attained at φ_1. Hence, from the Hölder inequality, we deduce that $\lambda_1 = \|\varphi_1\|^2/|\varphi_1|_2^2 > S/|\Omega|^{2/3}$.

Our main result reads as follows:

Theorem 1.1. *Let* $\lambda^* = \min\{\frac{S}{2(4|\Omega|)^{\frac{2}{3}}}, \frac{3^{2/3}a^{1/6}S^{1/2}}{b^{1/3}2^{19/6}|\Omega|^{2/3}}, \frac{3^{2/3}a^{1/3}}{16b^{2/3}|\Omega|^{2/3}}\}$.

(A1) *If* $\lambda_1 - \lambda^* < \lambda < \lambda_1$ *and* μ *is larger than some positive constant, then problem* (1.3) *has a pair of nontrivial solutions* $\pm u^\lambda$.

(A2) *If* $\lambda_k \le \lambda < \lambda_{k+1} = \cdots = \lambda_{k+m} < \lambda_{k+m+1}$ *for some* $k, m \in \mathbb{N}$ *and* $\lambda > \lambda_{k+1} - \lambda^*$ *and* μ *is larger than some positive constant, then problem* (1.1) *has* m *distinct pairs of nontrivial solutions* $\pm u_j^\lambda$, $j = 1, \ldots, m$.

The remainder of this paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we complete the proof of Theorem 1.1.

2. Preliminaries

Recall that a function $u \in H$ is called a weak solution of (1.3) if

$$(a + b\|u\|^2)\int_\Omega \nabla u \nabla v \mathrm{d}x = \lambda a \int_\Omega uv \mathrm{d}x + \mu b \int_\Omega |u|^2 uv \mathrm{d}x + \int_\Omega |u|^4 uv \mathrm{d}x, \quad \forall v \in H. \tag{2.1}$$