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Positive and negative solutions of one-dimensional beam equation

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ABSTRACT

In this paper, we show that the usual limitations on the coefficient $c = c(x)$ in the linear problem $u^{(4)}+c(x)u = h(x)$ with Navier boundary conditions and nonnegative right hand side h are not necessary to get the existence of positive or negative solutions whenever $c(x)$ is a nonconstant function.

1. Introduction and statement of the results

We study the existence of positive and negative solutions of the following problem

$$\begin{aligned} &u^{(4)} + c(x)u = h(x) \quad \text{in } (0,1), \\ &u(0) = u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{1}$$

with $c = c(x)$, $h = h(x)$ being continuous functions on $[0, 1]$ and $h(x) \geq 0$, $x \in [0, 1]$. Problems of these types arise in many applications. Let us mention, for example, nonlinear suspension bridge models introduced by Lazer and McKenna [1], where the problem (1) describes stationary behaviour of the bridge under a nonnegative loading (see also [2] and references therein). The positivity (or negativity) of the corresponding solution (i.e., the deflection of the roadbed) is crucial for the whole system and properties of possible non-stationary solutions. The coefficient $c(x)$ can be understood as a variable stiffness of the bridge ropes (cable stays). Even if we focus on one-dimensional ODE problem, we would like to mention works of Grunau and Sweers, where positive solutions for fourth order PDEs subject to different types of boundary conditions are investigated, see, e.g., [3,4].

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We consider the set

$$W := \{u \in C^4([0,1]) :\ u(0) = u(1) = u''(0) = u''(1) = 0\},$$

and the linear operator $L_c :\ W \to C([0,1])$ defined by

$$L_c u = u^{(4)} + c(x)u, \quad u \in W.$$

Then (1) is equivalent to the operator equation

$$L_c u = h.$$

We say that L_c is *strictly inverse positive* (SIP for short) on W if $u \in W$, $L_c u = h \gneqq 0$ in $[0,1]$ implies $u > 0$ in $(0,1)$ and, moreover, $u'(0) > 0$, $u'(1) < 0$. The definition of a *strictly inverse negative* (SIN for short) operator is similar.

For the sake of brevity we denote

$$c_m := \min_{x\in[0,1]} c(x) \quad \text{and} \quad c^m := \max_{x\in[0,1]} c(x),$$

and $c_0 := 4k_0^4$ with k_0 being the smallest positive solution of the equation $\tan k = \tanh k$ (i.e, $k_0 \approx 3.9266$ and $c_0 \approx 950.8843$). The classical solvability results state the following.

Proposition 1 (*cf. Usmani [5] and Yang [6]*). *Let $c(x) \neq -n^4\pi^4$ for any $n \in \mathbb{N}$ and all $x \in [0,1]$. Then the problem (1) has a unique classical solution $u \in W$. Moreover, if $-\pi^4 < c_m \leq 0$, then*

$$\|u\|_{C([0,1])} \leq \frac{\pi}{2(\pi^4 + c_m)} \|h\|_{C([0,1])}.$$

As for the SIP and SIN properties, we have the following results by Schröder [7] and Cabada, Cid and Sanchez [8].

Proposition 2 (*Schröder [7]*). *Let $-\pi^4 < c(x) \leq c_0$. Then L_c is SIP on W. Moreover, if $c(x) \equiv c$ (constant), then L_c is SIP on W if and only if $-\pi^4 < c \leq c_0$.*

Proposition 3 (*Cabada, Cid and Sanchez [8]*). *Let $-\frac{c_0}{4} \leq c(x) < -\pi^4$. Then L_c is SIN on W. Moreover, if $c(x) \equiv c$ (constant), then L_c is SIN on W if and only if $-\frac{c_0}{4} \leq c < -\pi^4$.*

Let us note that [8] treats only the case with constant c (see Proposition 2.1, Remark 2.1 and Proposition 3.1). The result for a nonconstant function $c = c(x)$ follows directly from Theorem 3.1 (V) of [8] with the choice $M = -c^m$ and the lower and upper solutions $\alpha, \beta \in W$, in reversed order $\beta \leq \alpha$ in $[0,1]$, given by

$$\alpha^{(4)} + c_m\alpha = h \quad \text{and} \quad \beta^{(4)} + c^m\beta = h. \tag{2}$$

Similarly, α and $\beta \in W$ defined by (2) are the lower and upper solutions of (1) also in the case $c(x) > -\pi^4$. This time, they are well ordered, i.e., $\alpha < \beta$, and Theorem 3.1 (II) of [8] with the choice $M = \max\{c^m, 0\}$ provides the alternative proof of the result of Proposition 2 for a nonconstant $c(x)$.

Our goal in this paper is to show that the conditions in Propositions 1–3 are necessary ones neither for the existence of the solution of (1), nor for its positivity or negativity, respectively. In particular, we prove the following statements.

Theorem 4. *Let $c, h \in C([0,1])$ be such that*

$$\int_{c<0} c(x)\,\mathrm{d}x > -4\pi^2. \tag{3}$$