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A theorem on energy integrals for linear second-order ordinary differential equations with variable coefficients

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ABSTRACT

We aim at demonstrating a novel theorem on the derivation of energy integrals for linear second-order ordinary differential equations with variable coefficients. Namely, in this context, we will present a possible and consistent method to overcome the traditional difficulty of deriving energy integrals for Lagrangian functions that explicitly exhibit the independent variable. Our theorem is such that it appropriately governs the arbitrariness of the variable coefficients in order to have energy integrals ensured. In view of the theoretical framework in which the theorem will be embedded, we will also demonstrate that it can be applied as a mathematical method to solve linear second-order ordinary differential equations with variable coefficients. These results are expected to have a generalized fundamental character.

1. Preliminaries and motivation

Consider the general linear second-order ordinary differential equation

$$\ddot{x} + a(t)\dot{x} + b(t)x + c(t) = 0, \tag{1}$$

where t is the independent variable, $a(t)$, $b(t)$ and $c(t)$ are the variable coefficients, and $x = x(t)$ is the unknown function. Overdot means differentiation with respect to the independent variable. In the realm of the inverse problem of Lagrangian mechanics (see, e.g., [1–3]), Eq. (1) is such that it results as Lagrange's equation of the extremum problem

$$\delta \int_{t_1}^{t_2} L(x, \dot{x}, t)dt = 0, \tag{2}$$

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where t_1 and t_2 are the definite limits, $\delta(.)$ is the usual operator of the calculus of variations, and the Lagrangian L is given by

$$L = M(t)\left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}b(t)x^2 - c(t)x\right), \tag{3}$$

with

$$M(t) = \exp\int^t a(\tau)d\tau. \tag{4}$$

The Hamiltonian H associated with Eq. (1) can be quickly obtained by inserting Eq. (3) into the classical identity $H = p\dot{x} - L$, with $p = \partial L/\partial\dot{x}$, i.e.

$$p = M(t)\dot{x}, \tag{5}$$

and so

$$H = M(t)\left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}b(t)x^2 + c(t)x\right). \tag{6}$$

Since $\partial L/\partial t$ does not vanish identically in the variational expression

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\dot{x} - L\right) = -\frac{\partial L}{\partial t} \tag{7}$$

(see Eq. (3)), it is not generally possible to state that the Hamiltonian H corresponds to an energy integral of Eq. (1). This is a well-known issue of classical analytical mechanics (see, e.g., [4, Chap. III], [5, Chap. X], [6, Chap. II], [7, Chap. V], [8, Chap. 1]). The upcoming content intends to propose a possible and consistent solution to this problem, namely we will demonstrate a theorem that consistently yields energy integrals for Eq. (1).

2. Theorem

Consider the following generalized transformation properties:

$$p \to \pi : p = M(t)^{1/2}\pi, \tag{8}$$

$$x \to \xi : \frac{1}{2}M(t)b(t)x^2 + M(t)c(t)x = \phi(\xi). \tag{9}$$

Suppose that this new variable $\xi = \xi(t)$ is the solution of the differential equation $\ddot{\xi} + \partial\phi(\xi)/\partial\xi = 0$. Noticing that $\ddot{\xi} + \partial\phi(\xi)/\partial\xi = 0$ is Lagrange's equation for the new Lagrangian $\mathcal{L} = \frac{1}{2}\dot{\xi}^2 - \phi(\xi)$, the corresponding new momentum π is naturally defined as $\pi = \partial\mathcal{L}/\partial\dot{\xi}$, i.e.

$$\pi = \dot{\xi}. \tag{10}$$

These definitions properly characterize the right-hand side of Eqs. (8) and (9). Then, if the coefficients of Eq. (1) satisfy the identity

$$\frac{1}{2}\left(a(t)b(t) + \dot{b}(t)\right)x^2 + \left(a(t)c(t) + \dot{c}(t)\right)x - \frac{1}{2}a(t)\dot{x}^2 = 0, \tag{11}$$

the transformation properties (8) and (9) are such that they transform the Hamiltonian H (6) into an energy integral of Eq. (1). This is the theorem.

Remark 1. Note that the condition (11) signifies a generalization of the classical notion of analytical mechanics which asserts that: if $\partial L/\partial t = 0$, then $L = L(x, \dot{x})$. The condition (11) mathematically imposes that $\partial L(x(t), \dot{x}(t), t)/\partial t = 0$ for $x = x(t)$ as the solution of Eq. (1). This *does not necessarily* imply $L = L(x, \dot{x})$.