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Regularity of the steering control for systems with persistent memory

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ABSTRACT

The following fact is known for large classes of distributed control systems: when the target is regular, there exists a regular steering control. This fact is important to prove convergence estimates of numerical algorithms for the approximate computation of the steering control.

In this paper we extend this property to a class of systems with persistent memory (of Maxwell/Boltzmann type) and we show that it is possible to construct such smooth control via the solution of an optimization problem.

1. Introduction

We study the following system where $x \in (0,\pi)$ and $t > 0$:

$$\begin{cases} w''(x,t) = w_{xx}(x,t) + \int_0^t M(t-s) w_{xx}(x,s)\,\mathrm{d}s, \quad w(0,t) = f(t), \quad w(\pi,t) = 0 \\ w(x,0) = 0, \ w'(x,0) = 0. \end{cases} \tag{1}$$

We assume $M(t) \in H^2(0,T)$ and $f(t) \in L^2(0,T)$ for every $T > 0$. As proved for example in [1], $w(x,t) \in C([0,T]; L^2(0,\pi)) \cap C^1([0,T]; H^{-1}(0,\pi))$ and, for every $(\xi,\eta) \in L^2(0,\pi) \times H^{-1}(0,\pi)$ and $T > 2\pi$, there exists $f \in L^2(0,T)$ such that $w(T) = \xi, w'(T) = \eta$. We prove:

Theorem 1. *The following properties hold:*

1. *Let $(\xi,\eta) \in H_0^1(0,\pi) \times L^2(0,\pi)$ and let $T > 2\pi$. There exists a steering control $f \in H_0^1(0,T)$.*

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2. *One of the smooth steering controls is the integral of the function g which realizes the minimum of a suitable quadratic functional introduced in Section 3.*

The statement 1 is proved in Section 2 while statement 2 is in Section 3.

We conclude this introduction with few comments. First we note that system (1) is often encountered in the study of viscoelasticity and diffusion equations with memory. When $M(t)=0$ of course it reduces to the string equation. In the case of the wave equation (even when x in regions of $\mathbb{R}^d$, $d>1$) Theorem 1 is known. The proof that we give, based on moment methods, shows in particular controllability (in $H_0^1(0,\pi)\times L^2(0,\pi)$) of the cascade connection of system (1) with an integrator. We refer the reader to [2, Ch. 11] and references therein for this idea and to [3] for a precise analysis of the reachable set using smooth controls in the case of the wave equation.

For memoryless systems, a result analogous to Theorem 1 is the key for a numerical analysis of the construction of steering controls via optimization methods, see [4].

Finally, it is easy to guess that Theorem 1 can be extended to the case $\dim x>1$ and to higher regularity degree of the target. This will be the subject of a future analysis.

2. The moment problem and the proof of Theorem 1 item 1

The following computations are a bit simplified if we integrate the first equation of (1) on $[0,t]$ and we write it in the equivalent form (here $N(t)=1+\int_0^t M(s)\,\mathrm{d}s$)

$$w'(x,t)=\int_0^t N(t-s)w_{xx}(x,s)\,\mathrm{d}s,\quad w(x,0)=0,\ w(0,t)=f(t),\ w(\pi,t)=0. \tag{2}$$

We use the orthonormal basis of $L^2(0,\pi)$ whose elements are $\Phi_n=\sqrt{(2/\pi)}\sin nx$, $n\in\mathbb{N}$, and we expand

$$w(x,t)=\sum_{n\in\mathbb{N}}\Phi_n(x)w_n(t),\quad w_n(t)=\sqrt{\frac{2}{\pi}}\int_0^\pi \Phi_n(x)w(x)\,\mathrm{d}x.$$

Then $w_n(x,t)$ must satisfy

$$w_n'(t)=-n^2\int_0^t N(t-s)w_n(s)\,\mathrm{d}s+n\int_0^t N(t-s)\left(\sqrt{2/\pi}f(s)\right)\,\mathrm{d}s.$$

The function $\sqrt{2/\pi}f$ will be renamed f.

Let $z_n(t)$ solve

$$z_n'(t)=-n^2\int_0^t N(t-s)z_n(s)\,\mathrm{d}s,\quad z_n(0)=1. \tag{3}$$

We have (see [5])

$$\begin{aligned} w_n(t)&=n\int_0^t\left(\int_0^{t-s}N(t-s-\tau)z_n(\tau)d\tau\right)f(s)\,\mathrm{d}s\\ &=\frac{1}{n}\int_0^t\left(\frac{\mathrm{d}}{\mathrm{d}s}z_n(t-s)\right)f(s)\,\mathrm{d}s, \qquad (4)\\ w_n'(t)&=n\int_0^t\left(-\frac{\mathrm{d}}{\mathrm{d}s}\int_0^{t-s}N(t-s-\tau)z_n(\tau)d\tau\right)f(s)\,\mathrm{d}s. \qquad (5)\end{aligned}$$

We require that a target $(\xi,\eta)\in H_0^1(0,\pi)\times L^2(0,\pi)$ is reached at time T, i.e. we require $(w(T),w'(T))=(\xi,\eta)$.