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Uniqueness of solution for boundary value problems for fractional differential equations☆

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ABSTRACT

In this work, we investigate the uniqueness of solutions for a class of nonlinear boundary value problems for fractional differential equations. The main novelty of this work is that the Lipschitz constant is related to the first eigenvalues corresponding to the relevant operators. Our analysis relies on the u_0-positive operator.

1. Introduction

In recent years, boundary value problems of nonlinear fractional differential equations have been studied extensively in the literature (see, for instance, [1–16] and their references). Most of the results told us that the fractional differential equations had at least single and multiple positive solutions by using techniques of nonlinear analysis. For example, the authors [9] considered the existence of multiple positive solutions for the following fractional differential equation with a negatively perturbed term

$$\begin{cases} -D_{0+}^{p}x(t) = p(t)f(t,x(t)) - q(t), \quad t \in (0,1), \\ x(0) = x'(0) = 0, \qquad x(1) = 0, \end{cases}$$

where D_{0+}^{p} is the standard Riemann–Liouville derivative, $2 < p \leq 3$ is a real number, $q : (0,1) \to [0,+\infty)$ is Lebesgue integrable and does not vanish identically on any subinterval of $(0,1)$. They established the existence results by Krasnosel'skii's fixed point theorem in a cone.

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However, few results can be found in the literature concerning the uniqueness of solutions for boundary value problems of fractional differential equations [17–20]. Rehman and Khan [19] studied the following multi-point boundary value problems

$$\begin{cases} D_t^{\alpha} y(t) = f(t, y(t), D_t^{\beta} y(t)), \quad t \in (0,1), \\ y(0) = 0, \qquad D_t^{\beta} y(1) - \sum_{i=1}^{m-2} \varsigma_i D_t^{\beta} y(\xi_i) = y_0, \end{cases}$$

where $1 < \alpha \leq 2$, $0 < \beta < 1$, $0 < \xi_i < 1 (i = 1, 2, \dots, m-2)$, $\varsigma_i \geq 0$ with $\sum_{i=1}^{m-2} \varsigma_i \xi_i^{\alpha-\beta-1} < 1$ and D_t^{α} represents the standard Riemann–Liouville fractional derivative. They obtained the uniqueness existence of solutions by means of the Banach fixed point theorem.

Motivated by the above works, we study the following boundary values problems for fractional differential equations to develop new uniqueness results

$$\begin{cases} D^p x(t) + p(t) f(t, x(t)) + q(t) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \quad x(1) = 0, \end{cases} \tag{1.1}$$

where $2 < p \leq 3$ is a real number. Under the assumption that $f(t, x)$ is a Lipschitz continuous function, by use of u_0-positive operator, we study the uniqueness existence of solution for the fractional differential equation (1.1). The interesting point is that the Lipschitz constant is related to the first eigenvalues corresponding to the relevant operators.

In the rest of this paper, we always suppose that the following assumptions hold:

(H_1) $p : (0,1) \to [0, +\infty)$ is continuous and does not vanish identically on any subinterval of $(0,1)$ such that

$$0 < \int_0^1 p(s) ds < +\infty.$$

(H_2) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_3) $q : (0,1) \to \mathbb{R}$ is continuous and Lebesgue integrable.

2. Preliminaries

For the convenience of the reader, we present here some necessary definitions from fractional calculus theory. These definitions and properties can be found in the recent monograph [1–5].

Definition 2.1. The Riemann–Liouville fractional integral of order $p > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2. The Riemann–Liouville fractional derivative of order $p > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D^p f(t) = \frac{1}{\Gamma(n-p)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{p-n+1}} ds,$$

where $n - 1 \leq \alpha < n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.