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Exponential stability of linear discrete systems with constant coefficients and single delay

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ABSTRACT

In the paper the exponential stability and exponential estimation of the norm of solutions to a linear system of difference equations with single delay

$$x(k+1) = Ax(k) + Bx(k-m), \quad k = 0, 1, \dots$$

is studied, where A, B are square constant matrices and $m \in \mathbb{N}$. New sufficient conditions for exponential stability are derived using the method of Lyapunov functions. Illustrative examples are given as well.

1. Preliminaries

Recently, the investigation of the properties (such as stability, representation of solutions, controllability) of linear difference systems with delay has been paid increasing attention [1–7]. The purpose of this paper is to give sufficient conditions for the exponential stability of linear difference systems with constant coefficients and single delay

$$x(k+1) = Ax(k) + Bx(k-m), \quad k = 0, 1, \dots \tag{1}$$

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where A, B are $n \times n$ constant matrices, $x = (x_1, \ldots, x_n)^T : \{-m, -m+1, \ldots\} \to \mathbb{R}^n$, and $m \in \mathbb{N}$ as well as to give an exponential estimate of the norm of solutions and compare the results with some previously published results. The Cauchy problem for the system (1) is defined by the state of the system on the whole interval of the solutions' previous history $x(k) = x_k$, $k = -m, -m+1, \ldots, 0$ where $x_k \in \mathbb{R}$. For an arbitrary matrix $\mathcal{B}$, we use the matrix norm $|\mathcal{B}| = (\lambda_{\max}(\mathcal{B}^T\mathcal{B}))^{1/2}$, and for vectors $x = (x_1, \ldots, x_n)^T$, the norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$. Moreover, denote by $\lambda_{\max}(\mathcal{A})$, $\lambda_{\min}(\mathcal{A})$ the maximum and minimum eigenvalues of a positive definite symmetric matrix $\mathcal{A}$. Also, we define $\varphi(\mathcal{A}) := \lambda_{\max}(\mathcal{A}) / \lambda_{\min}(\mathcal{A})$. For the basics of the stability theory of difference equations, we refer, e.g., to [8–10]. A trivial solution $x(k) = 0$, $k = -m, -m+1, \ldots$ of (1) is Lyapunov stable if, for arbitrary $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, for any other solution $x(k)$, we have $|x(k)| < \varepsilon$ for $k = 0, 1, \ldots$, and $\|x(0)\|_m < \delta(\varepsilon)$ where

$$\|x(0)\|_m := \max\{|x(i)|, i = -m, -m+1, \ldots, 0\}.$$

If, in addition $\lim_{k\to+\infty} |x(k)| = 0$, the trivial solution is called asymptotic stable. The trivial solution of system (1) is called Lyapunov exponentially stable if there exist constants $N > 0$ and $\theta \in (0, 1)$ such that

$$|x(k)| \le N\|x(0)\|_m \theta^k, \quad k = 1, 2, \ldots.$$

Note that, for homogeneous linear difference equations, stability of trivial solution is equivalent to stability of any other solution and the notions of asymptotic stability and exponential stability are equivalent to those of systems with constant coefficients.

The asymptotic stability of (1) can be studied by the corresponding characteristic equation

$$\det\left(\lambda^{m+1} I - \lambda^m A - B\right) = 0 \tag{2}$$

which, in general, is a polynomial equation of degree $(m+1)n$. A major difficulty in the investigation of the properties of roots of (2) is that, for large m, the dimension of the polynomial is high. The application of the well-known Schur–Cohn criterion [9] to verify that all roots of (2) lie inside the unit circle, is not easy because the computational difficulties grow substantially as the order of (2) increases. However, for the asymptotic stability of special classes of linear discrete equations with delay, simple sufficient and necessary criteria were derived, e.g. in [1]–[4]. The stability criteria derived in [1] are fully explicit with respect to the delay.

In the paper, we investigate the exponential stability of (1) by the second Lyapunov method. When this method is used, usually two of its modifications, motivated by the original ideas of the theory of differential equations with delays, are considered. The first is the method of finite-dimensional Lyapunov functions. The second one is the method of functionals of Lyapunov–Krasovskii. It is called the method of functionals because it is a discrete analog of the Lyapunov–Krasovskii method of functionals known in the theory of stability of functional differential equations with delay. In what follows, we use the former method.

Recall as well the known fact that, in the investigation of stability of linear discrete equations, an important role is played by what is called the Lyapunov equation

$$A^T H A - H = -C \tag{3}$$

where A is a given $n \times n$ matrix and H, C are $n \times n$ matrices, naturally arising when quadratic Lyapunov function $V(x) = x^T H x$ is utilized. It is known that the linear system $x(k+1) = Ax(k)$, $k = 0, 1, \ldots$ is exponentially stable, i.e. $\rho(A) < 1$, if and only if, for an arbitrary positive definite symmetric $n \times n$ matrix C, the matrix equation (3) has a unique solution—a positive definite symmetric matrix H [9].

Below we consider a similar approach to studying the stability of difference systems with delay (1). The rest of the paper is organized as follows. In Section 2, exponential stability of system (1) and exponential estimates are investigated. The concluding remarks and examples demonstrating the results obtained are considered in Section 3.