Applied Mathematics Letters 40 (2015) 13–16

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Integrability and generalized center problem of resonant singular point

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ARTICLE INFO

Article history:
Received 30 July 2014
Received in revised form 6 September 2014
Accepted 6 September 2014
Available online 17 September 2014

Keywords:
Generalized center
Integrability
Resonant singular point
Integrating factor

ABSTRACT

In this paper, integrability and generalized center condition of resonant singular point for a broad class of complex autonomous polynomial differential system are studied. A new method—integrating factor method of determining integrability of resonant singular point is obtained for any rational resonance ratio. At the same time, the relations of the first integral method and the integrating factor method with the normal form method are obtained.

1. Introduction

Recently, the classical center problem for real planar polynomial differential system

$$\frac{dx}{dt} = -y + P(x, y), \qquad \frac{dy}{dt} = x + Q(x, y), \tag{1}$$

where $P(x, y)$, $Q(x, y)$ are polynomials of x and y with degree no less than 2, is generalized by [1] to the system

$$\frac{dx}{dt} = px + P(x, y), \qquad \frac{dy}{dt} = -qy + Q(x, y), \tag{2}$$

which has a $p : -q$ resonant singular point at the origin. When P and Q are special polynomials and the resonance ratio is particular, integrability of system (2) has been studied by quite a few people (see, e. g., [1–10] for quadratic polynomials, [11–17] for cubic polynomials, [18,19] for quartic polynomials, and [20] for quintic polynomials) and there is no lack of excellent results such as [1,2,4].

In this paper, based on the previous works, we study further the integrability and generalized center condition of resonant singular point for complex polynomial differential system

$$\begin{cases} \dfrac{dz}{dT} = pz + \displaystyle\sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} z^{\alpha} w^{\beta} = Z, \\ \dfrac{dw}{dT} = -qw - \displaystyle\sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} w^{\alpha} z^{\beta} = -W, \end{cases} \tag{3}$$

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http://dx.doi.org/10.1016/j.aml.2014.09.005
0893-9659/

where z, w, T are independent complex variables, $a_{\alpha\beta}$, $b_{\alpha\beta}$ are complex constants, p, q are positive integers, and $(p, q) = 1$. In Section 2, we introduce some preliminary results. In Section 3, we introduce a new method—integrating factor method of determining integrability of resonant singular point and get another necessary and sufficient condition of determining integrability. What is more, we obtain the relations of the first integral method and the integrating factor method with the normal form method.

2. Some preliminary results

Lemma 2.1 (*See [21,4,5] etc.*). *We can derive successively the following formal series*

$$\begin{cases} u = z + \sum_{\alpha+\beta=2}^{\infty} A_{\alpha\beta} z^{\alpha} w^{\beta} \\ v = w + \sum_{\alpha+\beta=2}^{\infty} B_{\alpha\beta} w^{\alpha} z^{\beta}, \end{cases} \tag{4}$$

such that system (3) be transformed into its normal form:

$$\begin{cases} \frac{du}{dT} = pu \sum_{k=0}^{\infty} p_k (u^q v^p)^k \\ \frac{dv}{dT} = -qv \sum_{k=0}^{\infty} q_k (u^q v^p)^k, \end{cases} \tag{5}$$

with $p_0 = q_0 = 1$.

Definition 2.2 (*[5]*). We call $\mu_k = p_k - q_k$ "generalized singular point quantity of order k" at the origin of system (3). If $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0,\ \mu_k \neq 0$, then we call the origin a fine singular point of order k; If for all k, $\mu_k = 0$, then we call the origin a generalized center.

Remark. (i) If system (3) is a real system, then "μ_k" defined in Definition 2.2 is "the saddle point quantity of order k" defined in many other works (see e.g. [21,4] and references there in).

(ii) $\mu_0 = 0$ is a direct result of $p_0 = q_0 = 1$.

(iii) If $p = q = 1$, and $a_{\alpha\beta} = \overline{b_{\alpha\beta}}$, then under transformation $z = x + iy$, $w = x - iy$, $T = it$, $i = \sqrt{-1}$, system (3) becomes system (1). Systems (3) and (1) are concomitant each other. And the singular point quantities of system (3) are actually the focal values of system (1) except for possible constant factor which do not affect the order of singular point or focus.

Theorem 2.3 (*[5]*). *We can derive successively the following formal series*

$$F(z, w) = z^q w^p + h.o.t., \tag{6}$$

such that

$$\left.\frac{dF}{dT}\right|_{(3)} = \sum_{k=1}^{\infty} \lambda_k (z^q w^p)^{k+1}. \tag{7}$$

Here and throughout the paper 'h.o.t.' represents 'higher order terms'.

Theorem 2.4 (*[5]*). *The origin of system (3) is a generalized center if and only if there is an analytic first integral.*

3. Singular point quantity and integrability

Theorem 3.1. *The λ_k's appeared in Eq. (7) satisfy the following: if $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0,\ \mu_k \neq 0$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0,\ \lambda_k \neq 0,\ \lambda_k = pq\mu_k$ for all $k \geq 1$, and vice versa.*

Proof. Let $F = u^q v^p = z^q w^p + h.o.t.$, then from Eq. (4), normal form (5) and Eq. (7) we have

$$\begin{aligned} \left.\frac{dF}{dT}\right|_{(3)} = \left.\frac{dF}{dT}\right|_{(5)} &= pq \sum_{k=1}^{\infty} (p_k - q_k)(u^q v^p)^{k+1} = pq \sum_{k=1}^{\infty} \mu_k (u^q v^p)^{k+1} \\ &= pq \sum_{k=1}^{\infty} \mu_k [(z^q w^p)^{k+1} + h.o.t] = \sum_{k=1}^{\infty} \lambda_k (z^q w^p)^{k+1}. \end{aligned} \tag{8}$$

From Eq. (8) we can easily get that if $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0, \mu_k \neq 0$, then by mathematical induction we get $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0,\ \lambda_k \neq 0,\ \lambda_k = pq\mu_k$ for $k \geq 1$, and vice versa.