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Structural reliability analysis by univariate decomposition and numerical integration

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Abstract

This paper presents a new and alternative univariate method for predicting component reliability of mechanical systems subject to random loads, material properties, and geometry. The method involves novel function decomposition at a most probable point that facilitates the univariate approximation of a general multivariate function in the rotated Gaussian space and one-dimensional integrations for calculating the failure probability. Based on linear and quadratic approximations of the univariate component function in the direction of the most probable point, two mathematical expressions of the failure probability have been derived. In both expressions, the proposed effort in evaluating the failure probability involves calculating conditional responses at a selected input determined by sample points and Gauss–Hermite integration points. Numerical results indicate that the proposed method provides accurate and computationally efficient estimates of the probability of failure.

Keywords: Reliability; Probability of failure; Decomposition methods; Most probable point; Univariate approximation; Numerical integration

1. Introduction

A fundamental problem in time-invariant component reliability analysis entails calculation of a multi-fold integral [1–3]

$$P_F \equiv P\left[g(\boldsymbol{X}) < 0\right] = \int_{g(\boldsymbol{x})<0} f_{\boldsymbol{X}}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{1}$$

where $\boldsymbol{X} = \{X_1, \ldots, X_N\}^{\mathrm{T}} \in \mathbb{R}^N$ is a real-valued, N-dimensional ($N \geq 2$) random vector defined on a probability space $(\Omega, \mathcal{F}, P)$ comprising the sample space Ω, the σ-field $\mathcal{F}$, and the probability measure P; $g(\boldsymbol{x})$ is the performance function, such that $g(\boldsymbol{x}) < 0$ represents the failure domain; P_F is the probability of failure; and $f_{\boldsymbol{X}}(\boldsymbol{x})$ is the joint probability density function of $\boldsymbol{X}$, which typically represents loads, material properties, and geometry. The most common approach to compute the failure probability in Eq. (1) involves the first- and second-order reliability methods (FORM/SORM) [1–8], which are respectively based on linear (FORM) and quadratic (SORM) approximations of the limit-state surface at a most probable point (MPP) in the standard Gaussian space. When the distance β_{HL} between the origin and the MPP, a point on the limit-state surface that is closest to the origin, approaches infinity, FORM/SORM strictly provides asymptotic solutions. For non-asymptotic (finite β_{HL}) applications involving a highly nonlinear performance function, its linear or quadratic approximation may not be adequate and therefore resultant FORM/SORM predictions must be interpreted with caution [9]. In the latter cases, an importance sampling method developed by Hohenbichler and Rackwitz [10] can make the FORM/SORM result arbitrarily exact, but it may become expensive if a large number of costly numerical analyses, such as large-scale finite element analysis embedded in the performance function, are involved. Furthermore, the existence of multiple MPPs may lead to large errors in standard FORM/SORM approximations [3,8]. In that case, multi-point FORM/SORM, along with the system reliability concept, is required to improve component reliability analysis [8].

Recently, the authors have developed new decomposition methods, which can solve highly nonlinear reliability problems more accurately or more efficiently than FORM/SORM and

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$$y(\boldsymbol{v}) = \underbrace{\underbrace{\underbrace{y_0 + \sum_{i=1}^{N} y_i(v_i)}_{=\hat{y}_1(\boldsymbol{v})} + \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{N} y_{i_1 i_2}(v_{i_1}, v_{i_2})}_{=\hat{y}_2(\boldsymbol{v})} + \cdots + \sum_{\substack{i_1,\ldots,i_S=1 \\ i_1<\cdots<i_S}}^{N} y_{i_1\cdots i_S}(v_{i_1}, \ldots, v_{i_s})}_{=\hat{y}_S(\boldsymbol{v})} + \cdots + y_{12\cdots N}(v_1, \ldots, v_N),$$

Box I.

simulation methods [11,12]. A major advantage of these decomposition methods, so far based on the mean point [11] or MPP [12] of a random input as reference points, over FORM/SORM is that higher-order approximations of performance functions can be achieved using function values alone. In particular, an MPP-based univariate method developed in the authors' previous work involves univariate approximation of the performance function at the MPP, n-point Lagrange interpolation in the rotated Gaussian space, and subsequent Monte Carlo simulation [12]. The present work is motivated by an argument that the MPP-based univariate approximation, if appropriately cast in the rotated Gaussian space, permits an efficient evaluation of the component failure probability by multiple one-dimensional integrations.

This paper presents a new and alternative MPP-based univariate method for predicting the component reliability of mechanical systems subject to random loads, material properties, and geometry. Section 2 gives a brief exposition of a novel function decomposition at the MPP that facilitates a lower-dimensional approximation of a general multivariate function. Section 3 describes the proposed univariate method, which involves univariate approximation of the performance function at the MPP and univariate numerical integrations. Section 4 explains the computational effort and flowchart of the proposed method. Five numerical examples involving elementary mathematical functions and structural/solid-mechanics problems illustrate the method developed in Section 5. Comparisons have been made with alternative approximate and simulation methods to evaluate the accuracy and computational efficiency of the new method.

2. Multivariate function decomposition at MPP

Consider a continuous, differentiable, real-valued performance function $g(\boldsymbol{x})$ that depends on $\boldsymbol{x} = \{x_1, \ldots, x_N\}^{\mathrm{T}} \in \mathbb{R}^N$. If $\boldsymbol{u} = \{u_1, \ldots, u_N\}^{\mathrm{T}} \in \mathbb{R}^N$ is the standard Gaussian space, let $\boldsymbol{u}^* = \{u_1^*, \ldots, u_N^*\}^{\mathrm{T}}$ denote the MPP or beta point, which is the closest point on the limit-state surface to the origin. The MPP has a distance β_{HL}, which is commonly referred to as the Hasofer–Lind reliability index [1–3], determined by a standard nonlinear constrained optimization. Construct an orthogonal matrix $\boldsymbol{R} \in \mathbb{R}^{N\times N}$ whose Nth column is $\boldsymbol{\alpha}^* \equiv \boldsymbol{u}^*/\beta_{\mathrm{HL}}$, i.e., $\boldsymbol{R} = [\boldsymbol{R}_1 \mid \boldsymbol{\alpha}^*]$, where $\boldsymbol{R}_1 \in \mathbb{R}^{N\times N-1}$ satisfies $\boldsymbol{\alpha}^{*\mathrm{T}}\boldsymbol{R}_1 = \boldsymbol{0} \in \mathbb{R}^{1\times N-1}$. The matrix $\boldsymbol{R}$ can be obtained, for example, by Gram–Schmidt orthogonalization. For an orthogonal transformation $\boldsymbol{u} = \boldsymbol{R}\boldsymbol{v}$, let $\boldsymbol{v} = \{v_1, \ldots, v_N\}^{\mathrm{T}} \in \mathbb{R}^N$ represent the rotated Gaussian space with the associated MPP

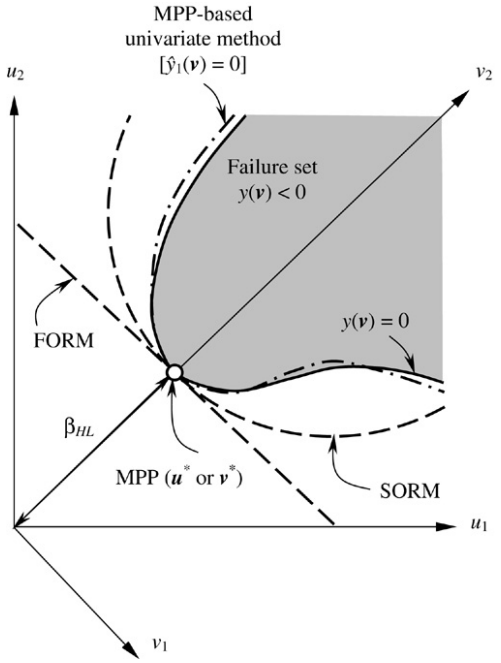

Fig. 1. Performance function approximations by various methods.

$\boldsymbol{v}^* = \{v_1^*, \ldots, v_{N-1}^*, v_N^*\}^{\mathrm{T}} = \{0, \ldots, 0, \beta_{\mathrm{HL}}\}^{\mathrm{T}}$. The transformed limit states $h(\boldsymbol{u}) = 0$ and $y(\boldsymbol{v}) = 0$ are therefore the maps of the original limit state $g(\boldsymbol{x}) = 0$ in the standard Gaussian space ($\boldsymbol{u}$ space) and the rotated Gaussian space ($\boldsymbol{v}$ space), respectively. Fig. 1 depicts FORM and SORM approximations of a limit-state surface at the MPP for $N = 2$.

Consider a decomposition of a general multivariate function $y(\boldsymbol{v})$, which can be viewed as a finite sum [11–13] (see Box I), where y_0 is a constant, $y_i(v_i)$ is a univariate component function representing an individual contribution to $y(\boldsymbol{v})$ by input variable v_i acting alone, $y_{i_1 i_2}(v_{i_1}, v_{i_2})$ is a bivariate component function describing the cooperative influence of two input variables v_{i_1} and v_{i_2}, $y_{i_1\cdots i_S}(v_{i_1}, \ldots, v_{i_S})$ is an S-variate component function quantifying the cooperative effects of S input variables $v_{i_1}, \ldots, v_{i_S}$, and so on. If

$$\hat{y}_S(\boldsymbol{v}) = y_0 + \sum_{i=1}^{N} y_i(v_i) + \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{N} y_{i_1 i_2}(v_{i_1}, v_{i_2}) + \cdots + \sum_{\substack{i_1,\ldots,i_S=1 \\ i_1<\cdots<i_S}}^{N} y_{i_1\cdots i_S}(v_{i_1}, \ldots, v_{i_s}) \tag{2}$$

represents a general S-variate approximation of $y(\boldsymbol{v})$, the univariate ($S = 1$) and bivariate ($S = 2$) approximations