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Sign-changing first derivative of positive solutions of forced second-order nonlinear differential equations

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ABSTRACT

Some oscillatory phenomena in physics, population, biomedicine and biochemistry are described by positive functions having sign-changing first derivative. Here, it is studied for all positive not necessarily periodic solutions of a large class of second-order nonlinear differential equations. It is based on a new reciprocal principle by which the classic oscillations of corresponding reciprocal linear equation causes the sign-changing first derivative of every positive solution of the main equation.

1. Introduction

Neutrino oscillation probabilities in particle physics (see [1, Chapter 4]), numbers of the predator and prey in a size-structured population (see [2, Section 2]), biomedical oscillations such as: cardiac, apnea, airway pressure (see [3,4]), and biochemical oscillations such as: glycolytic—two enzyme reactions, intracellular calcium (see [5, Sections 2, 4 and 9]), they all are modeled by positive real continuous functions $x(t)$ having sign-changing first derivative $x'(t)$ (see [1, Fig. 4.1], [2, Fig. 2], [3, Fig. 5], [4, Fig. 2], [5, Figs. 2.5, 2.7, 2.14, 4.31], [5, Figs. 9.2, 9.3, 9.8, 9.10, 9.21, 9.23]).

Under the positive $x(t)$ and sign-changing $x'(t)$ respectively, we mean as usual that $x(t) > 0$ on $[T, \infty)$ for some $T > 0$ and $(-1)^n x'(t) > 0$ on (a_n, b_n), $\forall n \in \mathbb{N}$, where $0 \le t_0 < a_1 < b_1 \le \cdots \le a_n < b_n \le a_{n+1} < b_{n+1} \le \cdots$, $a_n \to \infty$ as $n \to \infty$.

We study positive solutions $x(t)$ of the following second-order differential equation:

$$(r(t)x')' + p(t)x' + q(t)x + f(t, x) = e(t), \quad t \ge t_0, \tag{1.1}$$

where $p, e \in C([t_0, \infty), \mathbb{R})$, $r, q \in C^1([t_0, \infty), \mathbb{R})$, $x \in C^1([t_0, \infty), \mathbb{R}) \cap C^2((t_0, \infty), \mathbb{R})$, and $f(t, x)x \ge 0$ for all $t \ge t_0$, $x > 0$, and $e(t)$ is an oscillatory force, that is, $(-1)^n e(t) \ge 0$ on (a_n, b_n), $\forall n \in \mathbb{N}$.

The linear case $f(t, x) = x$ and the nonlinear case of Emden–Fowler type $f(t, x) = g(t)|x|^\nu \mathrm{sgn}(x)$, where $g(t) \ge 0$ and $\nu > 0$, are included in our main results too.

Any positive periodic smooth function $x(t)$ must have the sign-changing $x'(t)$. Existence of positive periodic solutions of the second-order differential equations with periodic coefficients have been studied in [6–9]. However, very often, $x(t)$ is not periodic and $x'(t)$ is sign-changing, see for instance the visual example given in Fig. 1.

Main problem. *Find sufficient condition on the coefficients $r(t)$, $p(t)$ and $q(t)$ such that every positive solution of Eq. (1.1) has the sign-changing first derivative.*

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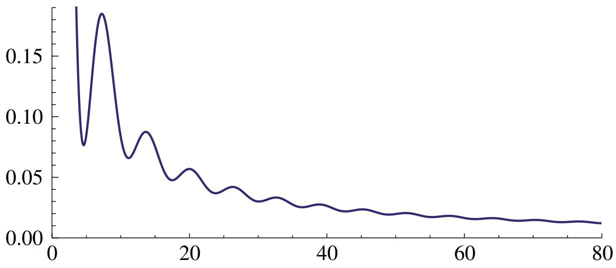

Fig. 1. Function $x(t) = 3t^{-2}\sin(t) + t^{-1}$.

Example 1.1. Equation $x'' + \frac{1}{4}t^{-2}x = (\frac{1}{4}t^{-2} - 1)\sin t$ has the nonoscillatory general solution $x(t) = \sqrt{t}(c_1 + c_2 \ln t) + \sin t$, $c_1^2 + c_2^2 > 0$, but $x'(t)$ is sign-changing. □

In contrast to the positive periodic behavior, the next three cases do not support the sign-changing $x'(t)$ for all positive $x(t)$: Eq. (1.1) may be oscillatory (and so, there is no any positive $x(t)$) or every positive solution is increasing (so, $x'(t)$ is not sign-changing, see [10]) or a kind of the coexistence occurs such as the following two: the simultaneously existence of positive, negative and sign-changing solutions, see for an abstract approach in [11], or two positive solutions $x_1(t) = 2t^{-1}$ and $x_2(t) = t^{-1}\sin(t) + 2t^{-1}$ of linear differential equation $x'' + 2t^{-1}x' + x = 2t^{-1}$ such that $x_1(t)$ is positive and increasing, but $x_2(t)$ is positive with sign-changing $x_2'(t) = -t^{-2}\big(2 - \cos(t)\big) + t^{-1}\cos(t)$.

To the best of our knowledge, there are only a few papers dealing, from different aspects, with solutions which have sign-changing first derivative: the nodal properties of $x(t)$ and $x'(t)$ (see [12,13]), the distance between zeros of $x(t)$ and $x'(t)$ (see [14,15]).

2. Preliminaries: Existence of solution of the reciprocal equation

The basic assumptions on the coefficients $r(t)$, $p(t)$, and $q(t)$ are the following:

$$r(t) > 0,\ q(t) > 0,\ p^2(t) \le 4r(t)q(t) \quad \text{and} \quad (-1)^n e(t) \ge 0 \text{ on } [a_n, b_n], \tag{2.1}$$

where $0 \le t_0 < a_1 < b_1 \le \cdots \le a_n < b_n \le a_{n+1} < b_{n+1} \le \cdots$, $a_n \to \infty$ as $n \to \infty$. The third inequality in (2.1) is not restrictive because it holds always in undamped case $p(t) \equiv 0$.

We start this section with a fundamental result on the existence of solutions of the first order ode's by sub-super solutions technique.

Lemma 2.1 (*[16, Theorem 1.2.1 or Theorem 1.1.4]*). *Let $F : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $\underline{\omega}, \bar{\omega} \in C^1([a, b], \mathbb{R})$ be two functions such that $\frac{d\underline{\omega}}{dt} \le F(t, \underline{\omega})$ and $\frac{d\bar{\omega}}{dt} \ge F(t, \bar{\omega})$ on (a, b). If $\underline{\omega}(t) \le \bar{\omega}(t)$ on $[a, b]$ and $\underline{\omega}(a) \le c_0 \le \bar{\omega}(a)$, then there exists a solution $\omega \in C^1([a, b], \mathbb{R})$ of the initial value problem $\frac{d\omega}{dt} = F(t, \omega)$ on (a, b) and $\omega(a) = c_0$, such that $\underline{\omega}(t) \le \omega(t) \le \bar{\omega}(t)$ on $[a, b]$.*

On the monotone iterative technique and sub-super solution method for the first order ode's, we refer reader to [17,18, 16,19], and references therein.

According to Lemma 2.1 and the assumption (2.1), we derive a sufficient condition for the interval global existence of a solution of the so-called *reciprocal linear differential equation:*

$$\left(\frac{1}{q(t)}y'\right)' - \frac{p(t)}{q(t)r(t)}y' + \frac{1}{r(t)}y = 0, \quad t \in (a_n, b_n),\ n \in \mathbb{N}, \tag{2.2}$$

which is associated to the main equation (1.1), where $y \in C([a_n, b_n], \mathbb{R}) \cap C^2((a_n, b_n), \mathbb{R})$.

Theorem 2.1. *Let $f(t, s)s \ge 0$ for all $t \ge t_0$, $s > 0$ and (2.1) hold. For every positive solution $x(t)$ of Eq. (1.1) such that $x'(t) \ne 0$ on $[a_{2n-1}, b_{2n-1}]$, $\forall n \ge n_0$ and some $n_0 \in \mathbb{N}$, there exists $\omega = \omega(t)$, $\omega \in C^1([a_{2n-1}, b_{2n-1}], \mathbb{R})$ which is a solution of equation:*

$$\begin{cases} \dfrac{d\omega}{dt} = q(t)\omega^2 + \dfrac{p(t)}{r(t)}\omega + \dfrac{1}{r(t)}, & t \in (a_{2n-1}, b_{2n-1}),\ n \ge n_0, \\ \omega(a_{2n-1}) = \dfrac{x(a_{2n-1})}{r(a_{2n-1})x'(a_{2n-1})}, & n \ge n_0. \end{cases} \tag{2.3}$$

Moreover, the function

$$y(t) = \exp\left(-\int_{a_{2n-1}}^{t} q(\tau)\omega(\tau)d\tau\right), \quad t \in [a_{2n-1}, b_{2n-1}],\ n \ge n_0, \tag{2.4}$$

satisfies the reciprocal equation (2.2) on the intervals (a_n, b_n) with odd n.