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Research announcement

Resonant Neumann problems with indefinite and unbounded potential

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ABSTRACT

In this paper, we report on some recent results obtained in our joint paper Papageorgiou and Rădulescu (in press). We establish multiplicity properties for a class of semilinear Neumann problems driven by the Laplacian plus on unbounded and indefinite potential. The reaction is a Carathéodory function which exhibits linear growth near $\pm\infty$. We allow for resonance to occur with respect to a nonprincipal nonnegative eigenvalue. The approach combines critical point theory, Morse theory and the Lyapunov–Schmidt method.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. Consider the following semilinear Neumann problem:

$$-\Delta u(z) + \beta(z)u(z) = f(z, u(z)) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{1}$$

Here $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. The potential function $\beta(\cdot)$ is in general unbounded and sign changing. More precisely, we assume that $\beta \in L^s(\Omega)$ with $s > N$. Also, the reaction $f(z, x)$ is a Carathéodory function that exhibits linear growth near $\pm\infty$. We allow for resonance to occur with respect to any nonnegative nonprincipal eigenvalue of $(-\Delta + \beta(\cdot),\ H^1(\Omega))$. So, we assume that asymptotically at $\pm\infty$ the quotient $\frac{f(z,x)}{x}$ is located in the spectral interval $[\hat{\lambda}_m, \hat{\lambda}_{m+1}]$ with $m \geqslant \max\{m_0, 2\}$, where $\hat{\lambda}_{m_0}$ is the first nonnegative eigenvalue of $(-\Delta + \beta(\cdot),\ H^1(\Omega))$. Hence, if $\beta \equiv 0$, then $m_0 = 2$ and so $m \geqslant 2$. We allow resonance with respect to the left end $\hat{\lambda}_m$ and nonuniform nonresonance with respect to the right end $\hat{\lambda}_{m+1}$. Problems with double resonance (that is, possible resonance at both ends of the spectral interval), were studied by O'Regan, Papageorgiou and Smyrlis [1], with $\beta \equiv 0$ (see also Hu and Papageorgiou [2] for Dirichlet problems with $\beta \neq 0$).

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The following linear eigenvalue problem has a central role in the analysis of problem (1):

$$-\Delta u(z)+\beta(z)u(z)=\lambda u(z)\quad \text{in } \Omega,\qquad \frac{\partial u}{\partial n}=0\quad \text{on } \partial\Omega. \tag{2}$$

This eigenvalue problem was studied by Papageorgiou and Smyrlis [3]. So, suppose that $\beta \in L^{N/2}(\Omega)$ if $N \geqslant 3$, $\beta \in L^r(\Omega)$ with $r > 1$ if $N = 2$ and $\beta \in L^1(\Omega)$ if $N = 1$. Let $\tau : H^1(\Omega) \to \mathbb{R}$ be the energy functional defined by

$$\tau(u)=\|Du\|_2^2+\int_\Omega \beta(z)u(z)^2dz\quad \text{for all } u\in H^1(\Omega).$$

Then the eigenvalue problem (2) has a smallest eigenvalue $\hat{\lambda}_1 > -\infty$ given by

$$\hat{\lambda}_1=\inf\left[\frac{\tau(u)}{\|u\|_2^2}:\ u\in H^1(\Omega),\ u\neq 0\right]. \tag{3}$$

From (3) it follows that we can find $\xi_0 > \max\{-\hat{\lambda}_1, 0\}$ such that

$$\tau(u)+\xi_0\|u\|_2^2\geqslant c_1\|u\|^2\quad \text{for all } u\in H^1(\Omega) \text{ and some } c_1>0. \tag{4}$$

Using (4) and the spectral theorem for compact self-adjoint operators (see, for example, Gasinski and Papageorgiou [4, p. 297]), we obtain a sequence $\{\hat{\lambda}_k\}_{k\geqslant 1}$ consisting of all the eigenvalues of (2) such that $\hat{\lambda}_k \to +\infty$ when $k \to \infty$. To these eigenvalues corresponds a sequence $\{\hat{u}_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ of eigenfunctions which form an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1(\Omega)$. Moreover, if $\beta \in L^s(\Omega)$ with $s > N$, then the regularity results of Wang [5], imply that $\{\hat{u}_n\}_{n\geqslant 1} \subseteq C^1(\overline{\Omega})$. These eigenvalues admit variational characterizations in terms of the Rayleigh quotient $\frac{\tau(u)}{\|u\|_2^2}$ for all $u \in H^1(\Omega) \setminus \{0\}$. In what follows, by $E(\hat{\lambda}_k)$, we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k,\ k \geqslant 1$.

Throughout this paper, our hypotheses on the potential function $\beta(\cdot)$ are the following:

H_0 : $\beta \in L^s(\Omega)$ with $s > N$ and $\beta^+ \in L^\infty(\Omega)$.

2. Existence of multiple solutions

We assume that the resonance occurs at $\pm\infty$ with respect to any nonnegative nonprincipal eigenvalue of $(-\Delta - \beta, H^1(\Omega))$. So, in what follows, $\hat{\lambda}_{m_0}$ denotes the first nonnegative eigenvalue of this operator.

The hypotheses on the reaction term $f(z, x)$ are the following:

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) there exist an integer $m \geqslant \max\{m_0, 2\}$ and a function $\eta \in L^\infty(\Omega)_+$ such that

$$\eta(z)\leqslant\hat{\lambda}_{m+1}\quad \text{a.e. in } \Omega,\ \eta\neq\hat{\lambda}_{m+1}$$
$$(f(z,x)-f(z,y))(x-y)\leqslant\eta(z)(x-y)^2\quad \text{for a.a } z\in\Omega,\ \text{all } x,y\in\mathbb{R};$$

(ii) $\hat{\lambda}_m \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x}$ uniformly for a.a. $z \in \Omega$;

(iii) if $F(z, x) = \int_0^x f(z, s)ds$, then we have

$$\lim_{x\to\pm\infty}[f(z,x)x-2F(z,x)]=-\infty\quad \text{uniformly for a.a. } z\in\Omega;$$

(iv) there exists a function $\vartheta \in L^\infty(\Omega)$ such that

$$\vartheta(z)\leqslant\hat{\lambda}_1\quad \text{a.e. in } \Omega,\ \vartheta\neq\hat{\lambda}_1$$
$$\limsup_{x\to 0}\frac{2F(z,x)}{x^2}\leqslant\vartheta(z)\quad \text{uniformly for a.a. } z\in\Omega;$$

(v) for every $\varrho > 0$, there exists $\xi_\varrho > 0$ such that

$$f(z,x)x+\xi_\varrho x^2\geqslant 0\quad \text{for a.a } z\in\Omega,\ \text{all } |x|\leqslant\varrho.$$

We observe that hypotheses H_1(i), (ii) imply that asymptotically at $\pm\infty$, the quotient $\frac{f(z,x)}{x}$ is in the spectral interval $[\hat{\lambda}_m, \hat{\lambda}_{m+1}]$ with possible resonance with respect to $\hat{\lambda}_m$ (see H_1(ii)), while at the other end we have nonuniform nonresonance (see H_1(i)).

The following function satisfies hypotheses H_1 above. For the sake of simplicity, we drop the z-dependence:

$$f(x)=\begin{cases}\vartheta x+\xi|x|^{p-2}x & \text{if } |x|\leqslant 1\\ \lambda x+\dfrac{c}{x} & \text{if } 1<|x|,\end{cases}$$

with $\vartheta < \hat{\lambda}_1$, $p > 2$, $\xi = \lambda + c - \vartheta$, $\lambda \in [\hat{\lambda}_m, \hat{\lambda}_{m+1})$ for some integer $m \geqslant \max\{m_0, 2\}$, $c > 0$, $2c < \lambda$.