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Stability and well posedness for a dissipative boundary condition with memory in electromagnetism

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ABSTRACT

In this paper the existence, uniqueness and asymptotic stability for an electromagnetic system with dissipative boundary conditions with memory are studied. For asymptotic stability it is crucial that the system satisfies the Second Law of Thermodynamics, which we will prove to be connected with the cosine Fourier transform of the kernel of the fading memory boundary condition.

1. Introduction

Among natural boundary conditions met within the context of differential systems, an outstanding role pertains to boundary conditions with memory studied in [1]. In electromagnetism, the Graffi–Schelkunoff boundary condition (see [2]) for harmonic fields has been extended and studied in [3–6] for fields $\mathbf{E}(x,t)$, $\mathbf{H}(x,t)$ satisfying the relation

$$\mathbf{E}_\tau(x,t) = \lambda_0(x)\mathbf{H}_\tau(x,t) \times \mathbf{n}(x) + \int_0^\infty \tilde{\lambda}(x,s)\mathbf{H}_\tau(x,t-s) \times \mathbf{n}(x)\,\mathrm{d}s \tag{1}$$

relative to the regular boundary $\partial\Omega$ with normal $\mathbf{n}$. Moreover, it was assumed that the coefficients λ_0 and $\tilde{\lambda}$ satisfy

$$\lambda_0(x) = -\int_0^\infty \tilde{\lambda}(x,s)\,\mathrm{d}s. \tag{2}$$

Boundary condition (1) exhibits both a stronger dissipation due to its first term $\lambda_0\mathbf{H}_\tau \times \mathbf{n}$ and a weaker one due to the past history of $\mathbf{H}_\tau$. Finally, it is to be observed that (2) weakens dissipativity when constant fields are involved. In this paper we consider a boundary condition weaker than (1) and very suitable from an electromagnetic point of view (see [7]), given by

$$\mathbf{E}_\tau(x,t) = \int_0^\infty \tilde{\lambda}(x,s)\mathbf{H}_\tau(x,t-s) \times \mathbf{n}(x)\,\mathrm{d}s. \tag{3}$$

Similarly, we shall show that even under condition (3) it is possible to prove asymptotic stability for an electromagnetic system consisting in a dielectric material whose dissipation stems solely from (3) as long as the kernel $\tilde{\lambda}$ satisfies the dissipative conditions following from the Second Law of Thermodynamics [8,9] and that we shall prove to be connected with the cosine Fourier transform of $\tilde{\lambda}$. The result on asymptotic stability also contains the proof of existence and uniqueness for the differential problem involved (see [10]).

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2. Electromagnetic system with a dissipative boundary condition

The electromagnetic field within a regular convex domain $\Omega \subset \mathbb{R}^3$ is described by the vectors electric field **E**, magnetic field **H**, electric displacement **D**, magnetic induction **B**, electric current **J**, and charge density ρ. Such fields satisfy the system of Maxwell equations in the space–time domain $Q = \Omega \times (0, T)$:

$$\frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}, \qquad \nabla \cdot \mathbf{D} = \rho \tag{4}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0. \tag{5}$$

In this paper we shall consider a dielectric and isotropic material for which

$$\mathbf{D}(x, t) = \epsilon(x)\mathbf{E}(x, t) \tag{6}$$

$$\mathbf{B}(x, t) = \mu(x)\mathbf{H}(x, t) \tag{7}$$

$$\mathbf{J}(x, t) = \mathbf{J}_0(x, t) \tag{8}$$

where ϵ and μ are scalar quantities according to the assumption of isotropy and represent the dielectric constant and the magnetic permittivity, respectively, of the electromagnetic medium. The field $\mathbf{J}_0$ represents an external impressed current. In the following we shall assume $\rho = 0$.

In this paper we shall assume the boundary of the domain Ω to consist in a good, albeit not perfect, conductor. For harmonic fields of frequency ω (see [1]) this implies a boundary condition of the following form

$$\mathbf{E}_\tau(x, \omega) = \lambda(x, \omega)\mathbf{H}_\tau(x, \omega) \times \mathbf{n}(x) \tag{9}$$

where $\mathbf{E}_\tau$ and $\mathbf{H}_\tau$ are the tangential components of **E** and **H** on the boundary $\partial\Omega$, while the scalar λ describes the constitutive properties of such boundary. Specifically, if λ approaches 0 then $\mathbf{E}_\tau = \mathbf{E} \times \mathbf{n}|_{\partial\Omega} = 0$ so that the boundary becomes a perfect conductor. The generalization for fields not necessarily of harmonic type was proposed in [3,7] using memory relation (1) together with restriction (2). Such relations imply that, for constant fields,

$$\mathbf{E}_\tau(x, t) = 0$$

again yielding a perfect conductor. This assumption may appear as too restrictive for some materials, which is why in this paper we propose the new weaker relation

$$\mathbf{E}_\tau(x, t) = \int_0^\infty \lambda(x, s)\mathbf{H}_\tau(x, t - s) \times \mathbf{n}(x)\, \mathrm{d}s \tag{10}$$

where the kernel $\lambda \in L^2(\Omega) \times H^1(0, \infty)$. Constitutive equation (10) for constant fields yields the relation

$$\mathbf{E}_\tau(x) = \left(\int_0^\infty \lambda(x, s)\, \mathrm{d}s\right)\mathbf{H}_\tau(x) \times \mathbf{n}(x) \tag{11}$$

suitable for describing a boundary consisting in a medium exhibiting moderate conductivity. Thus the electric field will be partly absorbed and partly reflected. Such conditions imply dissipation at the boundary. Therefore, following [7] we have that, for harmonic fields, Eq. (10) yields the relation

$$\mathbf{E}_\tau(x, \omega) = \hat{\lambda}(x, \omega)\mathbf{H}_\tau(x, \omega) \times \mathbf{n}(x) \tag{12}$$

where the cosine Fourier transform of the causal function $\lambda(x, t)$ is defined as

$$\hat{\lambda}_c(x, \omega) = \mathbb{R}\int_0^\infty \lambda(x, s)e^{-i\omega s}\, \mathrm{d}s = \int_0^\infty \lambda(x, s)\cos\omega s\, \mathrm{d}s. \tag{13}$$

The condition of dissipativity of an electromagnetic boundary is made precise in the following.

Definition 1. The boundary $\partial\Omega$ of an electromagnetic system is called *locally dissipative* if for all closed cycles of period T the inequality

$$\int_0^T \mathbf{E}_\tau(x, t) \times \mathbf{H}_\tau(x, t) \cdot \mathbf{n}(x)\, \mathrm{d}t > 0 \tag{14}$$

holds for all $x \in \partial\Omega$.