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On periodic solutions of subquadratic second order non-autonomous Hamiltonian systems ☆

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ABSTRACT

In this paper, we are concerned with the existence of periodic solutions for second order non-autonomous Hamiltonian systems under a new subquadratic growth condition. By using the minimax methods in critical point theory, an existence theorem is obtained, which extends and improves some known results in the literature.

1. Introduction and main result

Consider the second order systems

$$\begin{cases} -\ddot{u}(t) = \nabla F(t, u(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^{\mathbb{N}}$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \le a(|x|)b(t), \qquad |\nabla F(t, x)| \le a(|x|)b(t)$$

for all $x \in \mathbb{R}^{\mathbb{N}}$ and a.e. $t \in [0, T]$.

It follows from assumption (A) that the corresponding function φ on H_T^1 given by

$$\varphi(u) := \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \int_0^T F(t, u(t))dt$$

is continuously differentiable and weakly lower semicontinuous on H_T^1, where

$$H_T^1 := \{u : [0, T] \to \mathbb{R}^{\mathbb{N}} \,|\, u \text{ is absolutely continuous}, u(0) = u(T), \dot{u} \in L^2(0, T; \mathbb{R}^{\mathbb{N}})\}$$

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is a Hilbert space with the norm defined by

$$\|u\| := \left(\int_0^T |u(t)|^2 dt + \int_0^T |\dot{u}(t)|^2 dt\right)^{1/2}$$

for each $u \in H_T^1$. Moreover,

$$(\varphi'(u), v) = \int_0^T (\dot{u}(t), \dot{v}(t))dt - \int_0^T (\nabla F(t, u(t)), v(t))dt$$

for all $u, v \in H_T^1$. It is well known that the solutions of problem (1.1) correspond to the critical point of φ.

Considerable attention has been paid to the periodic solutions of problem (1.1) in the past decades. We refer the reader to [1–15] and the reference therein. Specially, in [1], Tang and Wu established the existence of periodic solutions for problem (1.1) when potential $F(t, x)$ was subquadratic. Concretely speaking, they obtained the following theorem:

Theorem A (*Tang and Wu [1]*). *Suppose that F satisfies assumption* (A) *and the following conditions:*

(S_1) *There exists $0 < \mu < 2$, $M_1 > 0$ such that*

$$(\nabla F(t, x), x) \leq \mu F(t, x)$$

for all $|x| \geq M_1$ and a.e. $t \in [0, T]$;

(S_2) $F(t, x) \to +\infty$ *as $|x| \to +\infty$ uniformly for a.e. $t \in [0, T]$.*

Then problem (1.1) has at least one solution in H_T^1.

In the present paper, we will further study the existence of periodic solutions for problem (1.1) under a new subquadratic growth condition instead of (S_1). For the sake of convenience, in the sequel, $\mathcal{H}$ will denote the space of continuous function space such that, for any $\theta \in \mathcal{H}$, there exists constant $M_2 > 0$ such that

(i) $\theta(t) > 0 \quad \forall t \in \mathbb{R}^+$,

(ii) $\displaystyle\int_{M_2}^t \frac{1}{s\theta(s)} ds \to +\infty \quad \text{as } t \to +\infty.$

Now, we are ready to state our main result of this paper.

Theorem 1.1. *Assume that F satisfies assumption* (A) *and the following conditions:*

(H_1) *There exists $\theta(|x|) \in \mathcal{H}$ with $0 < \frac{1}{\theta(|x|)} < 2$, $M_2 > 0$ such that*

$$(\nabla F(t, x), x) \leq \left(2 - \frac{1}{\theta(|x|)}\right) F(t, x)$$

for all $|x| \geq M_2$ and a.e. $t \in [0, T]$;

(H_2) $F(t, x) \geq 0$ *as $|x| \to +\infty$ uniformly for a.e. $t \in [0, T]$;*

(H_3) $\int_0^T \frac{F(t,x)}{\theta(|x|)} dt \to +\infty$ *as $|x| \to +\infty$.*

Then problem (1.1) has at least one solution in H_T^1.

Remark 1.1. Put $\inf_{|x| \geq M_2} \frac{1}{\theta(|x|)} := k$, where k is a constant, we see that

(a) Compared with Theorem 1.1, under the assumption (H_2), if $k > 0$, then (H_1) and (S_1) are equivalent, however, (H_1) is weaker than (S_1) when $k = 0$.

(b) Theorem 1.1 generalizes Theorem A even if $k > 0$. In fact, by (H_2), when $k > 0$, (H_3) is nothing but

(H_3^*) $\int_0^T F(t, x)dt \to +\infty$ as $|x| \to +\infty$,

while, (H_2) and (H_3^*) are much weaker than (S_2).

(c) There are functions $F(t, x)$ satisfying our Theorem 1.1 and not satisfying the results in [1–15]. For example, let

$$F(t, x) = k(t)\frac{2 + |x|^2}{\ln(2 + |x|^2)}, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^{\mathbb{N}},$$

where

$$k(t) = \begin{cases} \sin \dfrac{2\pi t}{T}, & t \in [0, T/2], \\ 0, & t \in [T/2, T]. \end{cases}$$

Setting $\theta(|x|) = \ln(2 + |x|^2)$, a straightforward computation shows that F satisfies the conditions (H_1)–(H_3) of Theorem 1.1, but it does not satisfy the corresponding conditions of Theorem A.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and present some preliminary results that will be used for the proof of Theorem 1.1. In Section 3, we are devoted to prove our main result.