Applied Mathematics Letters 35 (2014) 7–11

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Positive solutions for Neumann problems with indefinite and unbounded potential

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ARTICLE INFO

Article history:
Received 24 February 2014
Accepted 3 April 2014
Available online 13 April 2014

Keywords:
Positive solutions
Maximum principle
Picone's identity
Indefinite and unbounded potential
Logistic equation

ABSTRACT

We consider semilinear Neumann equations with an indefinite and unbounded potential. We establish the existence and uniqueness of positive solutions. We show that our setting incorporates as special cases several parametric equations of interest (such as the equidiffusive logistic equation).

1. Introduction

In this paper we deal with the following semilinear Neumann problem

$$-\Delta u(z)+\beta(z)u(z)=f(z,u(z)) \text{ in } \Omega, \quad \frac{\partial u}{\partial n}=0 \text{ on } \partial\Omega. \tag{1}$$

Here $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary, β is a potential function which is in general unbounded and sign changing. The reaction $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \longmapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \longmapsto f(z,x)$ is continuous), which exhibits general growth conditions near $+\infty$ and near 0^+. As we will see these conditions incorporate in our framework as special cases various parametric problems such as equidiffusive logistic equations. We are interested in the existence and uniqueness of positive solutions.

Recently semilinear Neumann problems with unbounded and indefinite potential were studied by Papageorgiou and Rădulescu [1]. They deal with equations in which the reaction $f(z,x)$ exhibits an asymmetric behavior at $+\infty$ and at $-\infty$ (jumping nonlinearity) and they prove multiplicity theorems providing sign information for all the solutions. We mention also the recent works of Mugnai and Papageorgiou [2], who examine equations driven by the p-Laplacian plus an indefinite potential and of Papageorgiou and Smyrlis [3], who consider a special class of coercive semilinear equations.

2. Positive solutions

In the analysis of problem (1) we will use the Sobolev space $H^1(\Omega)$ and the ordered Banach space $C^1(\overline{\Omega})$. The positive cone of the latter space is given by

$$C_+=\{u \in C^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

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http://dx.doi.org/10.1016/j.aml.2014.04.006
0893-9659/

This cone has a nonempty interior

$$\operatorname{int} C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

The hypotheses on the potential function $\beta(\cdot)$ are the following.

$H(\beta)$: $\beta \in L^s(\Omega)$ with $s > \frac{N}{2}$ and $\beta^+ \in L^\infty(\Omega)$

By $\{\hat{\lambda}_n\}_{n\geqslant 1}$ we denote the distinct eigenvalues of the differential operator $u \to -\Delta u + \beta(z)u$, $u \in H^1(\Omega)$. We know that $\hat{\lambda}_1$ is simple and

$$\hat{\lambda}_1 = \inf\left[\frac{\xi(u)}{\|u\|_2^2} : u \in H^1(\Omega),\ u \neq 0\right] \tag{2}$$

where $\xi(u) = \|Du\|_2^2 + \int_\Omega \beta(z)u^2dz$ for all $u \in H^1(\Omega)$. The infimum in (2) is realized on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1$. From (2) it is clear that the elements of this eigenspace do not change sign. By $\hat{u}_1$ we denote the positive L^2-normalized (that is, $\|\hat{u}_1\|_2 = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. The regularity results of Wang [4] imply that $\hat{u}_1 \in C_+\backslash\{0\}$ and in fact using $H(\beta)$ and the maximum principle of Vazquez [5], we have $\hat{u}_1 \in \operatorname{int} C_+$. The next lemma is a consequence of these properties (see Papageorgiou and Rădulescu [1]).

Lemma 1. *If $\vartheta \in L^\infty(\Omega)$ and $\vartheta(z) \leqslant \hat{\lambda}_1$ a.e. in Ω, $\vartheta \not\equiv \hat{\lambda}_1$, then there exists $c_0 > 0$ such that*

$$\xi(u) - \int_\Omega \vartheta(z)u^2dz \geqslant c_0\|u\|^2 \quad \text{for all } u \in H^1(\Omega).$$

The hypotheses on the reaction $f(z, x)$ are as follows.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leqslant a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$ with $a \in L^\infty(\Omega)_+$, $2 < r < 2^* = \begin{cases} \frac{2N}{N-2} & \text{if } 3 \leqslant N \\ +\infty & \text{if } N = 1, 2 \end{cases}$;

(ii) $\limsup_{x\to+\infty} \frac{f(z,x)}{x} \leqslant \vartheta(z)$ uniformly for a.a. $z \in \Omega$, with $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant \hat{\lambda}_1$ a.e in Ω, $\vartheta \not\equiv \hat{\lambda}_1$;

(iii) $\liminf_{x\to 0^+} \frac{f(z,x)}{x} \geqslant \eta(z)$ uniformly for a.a. $z \in \Omega$, with $\eta \in L^\infty(\Omega)$, $\eta(z) \geqslant \hat{\lambda}_1$ a.e. in Ω, $\eta \not\equiv \hat{\lambda}_1$.

Remark 1. Since we are interested in positive solutions and the above hypotheses concern the positive semi-axis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$. Note that according to hypothesis $H(f)$ (i), $f(z, \cdot)$ has subcritical growth. Finally hypotheses $H(f)$ (ii), (iii) imply that the quotient $\frac{f(z,x)}{x}$ crosses at least the principal eigenvalue $\hat{\lambda}_1$ as we move from 0^+ to $+\infty$.

From the spectral analysis of Papageorgiou and Rădulescu [1] we know that there exists $\gamma_0 > \max\{-\hat{\lambda}_1, 1\}$ such that

$$\xi(u) + \gamma_0\|u\|_2^2 \geqslant c_1\|u\|^2 \quad \text{for all } u \in H^1(\Omega),\ \text{some } c_1 > 0. \tag{3}$$

We introduce the Carathéodory function $\hat{f}(z, x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ f(z, x) + \gamma_0 x & \text{if } 0 < x \end{cases}$ and its primitive $\hat{F}(z, x) = \int_0^x \hat{f}(z, s)ds$.

Proposition 2. *If hypotheses $H(\beta)$ and $H(f)$ hold, then problem (1) has at least one positive solution $u_0 \in \operatorname{int} C_+$.*

Proof. Let $\hat{\varphi} : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\hat{\varphi}(u) = \frac{1}{2}\xi(u) + \frac{\gamma_0}{2}\|u\|_2^2 - \int_\Omega \hat{F}(z, u)dz \quad \text{for all } u \in H^1(\Omega).$$

Hypotheses $H(f)$ (i), (ii) imply that given $\epsilon > 0$, we can find $c_2 = c_2(\epsilon) > 0$ such that

$$F(z, x) \leqslant \frac{1}{2}(\vartheta(z) + \epsilon)(x^+)^2 + c_2 \quad \text{for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R} \tag{4}$$

(recall that for all $x \in \mathbb{R}$, $x^\pm = \max\{\pm x, 0\}$). Then

$$\hat{\varphi}(u) \geqslant \frac{1}{2}\left[\xi(u^+) - \int_\Omega \vartheta(z)(u^+)^2dz\right] - \frac{\epsilon}{2}\|u^+\|^2 + \frac{1}{2}\xi(u^-) + \frac{\gamma_0}{2}\|u^-\|_2^2 - c_2|\Omega|_N \quad (\text{see } (4)),$$

where $u^\pm(\cdot) = u(\cdot)^\pm$ and $|\cdot|_N$ denotes the Lebesgue measure on $\mathbb{R}^N$. Using Lemma 1 and (3), we obtain

$$\hat{\varphi}(u) \geqslant \frac{c_0 - \epsilon}{2}\|u^+\|^2 + \frac{c_1}{2}\|u^-\|^2 - c_2|\Omega|_N.$$

Choosing $\epsilon \in (0, c_0)$, we see that $\hat{\varphi}$ is coercive. Also, using the Sobolev embedding theorem, we check that $\hat{\varphi}$ is sequentially weakly lower semicontinuous. So, we can find $u_0 \in H^1(\Omega)$ such that

$$\hat{\varphi}(u_0) = \inf[\hat{\varphi}(u) : u \in H^1(\Omega)], \tag{5}$$