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Global regularity for the incompressible 2D generalized liquid crystal model with fractional diffusions

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ABSTRACT

In this paper, we consider the global regularity for the 2D generalized liquid crystal model with the fractional diffusion term $-\Lambda^{2\alpha}u$ for the velocity field and $-\Lambda^{2\beta}d$ for the vector field modeling the orientation of the crystal molecules. Global existence of smooth solutions is proved for the case $\alpha = 0,\ \beta > 1$.

1. Introduction

The 2D generalized liquid crystal model reads:

$$u_t + u\cdot\nabla u + \nabla p + \Lambda^{2\alpha}u = -\nabla d\cdot\Delta d, \tag{1.1}$$

$$d_t + u\cdot\nabla d + \Lambda^{2\beta}d = -f(d), \tag{1.2}$$

$$\nabla\cdot u = 0, \tag{1.3}$$

$$(u, d)(x, 0) = (u_0, d_0)(x), \tag{1.4}$$

where $u(x, t) \in \mathbb{R}^2$ is the velocity field, $d(x, t) \in \mathbb{R}^2$ is a vector field modeling the orientation of the crystal molecules, p is a scalar pressure, while $\alpha \geq 0,\ \beta \geq 0$ are real parameters. $f(d) := (|d|^2 - 1)d$ and the operator $\Lambda = (-\Delta)^{\frac{1}{2}}$ is defined by $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$; here $\widehat{f}$ denotes the Fourier transform of f. We identify the case $\alpha = 0$ as the 2D generalized liquid crystal model with zero velocity diffusion. When $\alpha = 0, \beta = 1$, the system is a simplified version of the Ericksen–Leslie system modeling the hydrodynamics of nematic liquid crystals developed during the period of 1958 through 1968 [1–3]. We notice that if $d \equiv 0, \alpha = 1$, then system (1.1)–(1.4) becomes to the Navier–Stokes equations. So, to study system (1.1)–(1.4) is valuable and interesting in both mathematical and physical sense.

Now, we mention some known results about the system. When $\alpha = \beta = 1$, the existence and uniqueness of the weak and smooth solutions for system (1.1)–(1.4) is given in [4–6]. Local existence of classical solutions for the nematic liquid crystal

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flows was established in [7]. Later, Zhou established a regularity criterion for it as $\int_0^T \frac{\|\nabla u\|_{L^p}^r}{1+\ln(e+\|\nabla u\|_{L^p})} dt < +\infty$ with $\frac{2}{r}+\frac{3}{p}=2$, $2 \le p \le 3$ in [8]. Recently, in [9], Fan, Nakamura and Zhou established global regularity for this system with mixed partial viscosity. The global strong solution to the density-dependent 2D liquid crystal flows was studied in [10]. Moreover, some regularity criteria are proved for the system with zero dissipation in [11].

This paper focuses on the system with $\alpha = 0,\ \beta > 1$. From the study of the 2D incompressible generalized MHD equations (refer [12–15]), we know that to give the global wellposedness of classical solution for the system with $\alpha = 0,\ \beta > 1$ is difficult and important. Our main result is the following global regularity criteria.

Theorem 1.1. *Assume $\alpha = 0,\ \beta > 1$, and $(u_0, d_0)(x) \in H^2(\mathbb{R}^2) \times H^3(\mathbb{R}^2)$, then the 2D liquid crystal model (1.1)–(1.4) has a unique global classical solution $(u, d)(x, t)$ satisfying*

$$u \in L^\infty(0, T; H^2(\mathbb{R}^2)),$$
$$d \in L^\infty(0, T; H^3(\mathbb{R}^2)),\ d \in L^2(0, T; H^{3+\beta}(\mathbb{R}^2)).$$

Remark 1.1. Theorem 1.1 is partially motivated by the recent works on 2D incompressible generalized MHD equations and the liquid crystal model (refer [12–16] for details). To establish global regularity criteria for system (1.1)–(1.4) with $\alpha > 0,\ \beta = 1$ or $\alpha + \beta > 1$ are open problems, and these are also our further work.

Now, we list some notations that will be used in our paper. We use the concise notations $L^{p,q}$ and ∂_i to denote the spaces $L^p(0, T; L^q(\mathbb{R}^2))$ for the fixed positive number T and the partial derivative to the ith space variable respectively. Throughout this paper, C denotes a generic positive constant (generally large); it may be different from line to line. Use $\hat{f}$ to denote the Fourier transform of f.

2. Key lemmas

Before giving the proof of our main theorem, we give two lemmas which are crucial in the proof of our theorem. First, let us consider the following equation:

$$v_t + \Lambda^{2\beta} v = f, \tag{2.1}$$
$$v(x, 0) = v_0(x). \tag{2.2}$$

Similar to the heat equation, we get

$$v(x,t) = \int_{\mathbb{R}^2} t^{-\frac{1}{\beta}} h\left(\frac{x-y}{t^{\frac{1}{2\beta}}}\right) v_0(y) dy + \int_0^t \int_{\mathbb{R}^2} (t-s)^{-\frac{1}{\beta}} h\left(\frac{x-y}{(t-s)^{\frac{1}{2\beta}}}\right) f(y,s) dy ds, \tag{2.3}$$

where $h(x) = (e^{-|\cdot|^{2\beta}})^\vee(x)$. Now, we give some estimation for h. As a direct calculation we can deduce the following L^2 estimation for ∇h.

Lemma 2.1. *For any $\beta > 0$, we have*

$$\|\nabla h\|_{L^2} \le C, \tag{2.4}$$

where C is a positive constant which only depends on β.

Proof.

$$\begin{aligned} \|\nabla h\|_{L^2} &= \|\widehat{\nabla h}\|_{L^2} = \int_{R^2} i\xi\widehat{h} \cdot \overline{i\xi\widehat{h}} d\xi = \int_{R^2} |\xi|^2 \widehat{h}\overline{\widehat{h}} d\xi \\ &= \int_{R^2} |\xi|^2 e^{-2|\xi|^{2\beta}} d\xi = C\int_0^\infty e^{-2r^{2\beta}} r^3 dr \\ &= C\,\Gamma\left(\frac{2}{\beta}\right). \end{aligned}$$

Here $\Gamma(s), s > 0$ is the Gamma Function. □

The following lemma is proved in [15].

Lemma 2.2 ([15]). *Let l be a nonnegative integer and $\eta \ge 0$, then*

$$\|\nabla^l h\|_{L^1} + \|\Lambda^\eta h\|_{L^1} \le C, \tag{2.5}$$

where C is a positive constant.

3. Proof of Theorem 1.1

First, we give the following priori estimates.