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Existence of positive solutions for a dynamic equation on measure chains[☆]

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ABSTRACT

In this paper we consider the following dynamic equation on a measure chain T

$$Lx(t) = -x^{\Delta\Delta}(t) + p(t)x^{\Delta}(t) = f(t, x(\sigma(t))), \quad t \in [a, b],$$

with the boundary value conditions $x(a) = 0 = x(\sigma^2(b))$. Unlike many other dynamic equations on measure chains, it involves the term $x^{\Delta}(t)$, so it is hard to get Green's function. We obtain Green's function of this equation. And the Leray–Schauder fixed point theorem will be used to prove the existence of at least one solution.

1. Introduction

For a long time, boundary value problems for dynamic equations on measure chains have attracted much increasing attention (see, for instance, [1–5] and the references therein). The authors studied the existence of solutions of those equations, however, most of those equations are very special (see [6–11]). In particular, C.J. Chyan and P.J.Y. Wong discussed the following system on a time scale T in [12]:

$$y^{\Delta\Delta}(t) + P(t, y(\sigma(t))) = 0, \quad t \in [a, b].$$
$$y(a) = 0 = y(\sigma^2(b)).$$

By using different fixed point theorems, criteria are established for the existence of three positive solutions of the boundary value problem. And H. Luo (see [13]) discussed

$$u^{\Delta\Delta}(t) + f(t, u(\sigma(t))) = 0, \quad t \in [0, T].$$
$$u(0) = 0 = u(\sigma^2(T)).$$

The author gave a global description of the branches of positive solutions for the dynamic equation. Similar equations can be seen in [14,15].

Here we discuss the boundary value problem for the following equation of the general form:

$$-x^{\Delta\Delta}(t) + p(t)x^{\Delta}(t) = f(t, x(\sigma(t))), \quad t \in [a, b].$$
$$x(a) = 0 = x(\sigma^2(b)).$$

The rest of the paper is organized as follows. In Section 2, we present some definitions and theorems on measure chains. In Section 3, we discuss Green's function and the existence of a solution for this equation.

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2. Preliminary

In this section we will introduce several definitions and theorems on a measure chain. More details can be seen in [16–20]. A measure chain (time scale) is an arbitrary nonempty closed subset of the real numbers R.

Definition 2.1. Let T be a measure chain and define the forward jump operator σ on T by

$$\sigma(t) := \inf\{s > t : s \in T\} \in T,$$

for all $t \in T$ and the backward jump operator ρ on T by

$$\rho(t) := \sup\{s < t : s \in T\} \in T,$$

for all $t \in T$. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$, we say t is right-dense, while if $\rho(t) = t$ we say t is left-dense. $\mu : T \to [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

Throughout the paper we make the blanket assumption that $a \le b$ are points in T.

Definition 2.2. Define the interval $[a, b]$ in T by

$$[a, b] := \{t \in T : a \le t \le b\},$$

other types of intervals are defined similarly. The set T^k is derived from T as follows: if T has a left scattered maximum m, then $T^k = T - \{m\}$. Otherwise, $T^k = T$.

Definition 2.3. Assume $f : T \to R$ and let $t \in T^k$, then we define $f^{\Delta}(t)$ to be the number (provided it exists) with property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $f^{\Delta}(t)$ the delta derivative of $f(t)$ and it turns out that f^{Δ} is the usual derivative if $T = R$ and is the usual forward difference operator if $T = Z$.

Definition 2.4. If $p \in \Re$, then we define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right).$$

Here $\xi_h(z) = \frac{1}{h}\log(1 + zh)$, where "log" is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in C$.

Lemma 2.1 (*[21, Theorem 3.3], Leray–Schauder Fixed Point Theorem*)**.** *Assume E is a Banach space and $A : E \to E$ is completely continuous. If the set $S = \{x | x \in E, x = \lambda Ax, 0 < \lambda < 1\}$ is bounded, then A has a fixed point in E.*

3. Main results

We consider the following boundary value problem on a measure chain

$$-x^{\Delta\Delta}(t) + p(t)x^{\Delta}(t) = f(t, x(\sigma(t))), \quad t \in [a, b], \tag{3.1}$$

$$x(a) = 0, \qquad x(\sigma^2(b)) = 0. \tag{3.2}$$

Motivated by [1], we begin with the following definition.

Definition 3.1. We say $x(t, s)$ is the Cauchy function for $Lx = 0$ provided for each fixed $s \in T$, $x(t, s)$ is the solution of the IVP

$$Lx(t, s) = 0, \qquad x(\sigma(s), s) = 0, \qquad x^{\Delta}(\sigma(s), s) = -1.$$

The Cauchy function for $Lx = -x^{\Delta\Delta}(t) + p(t)x^{\Delta}(t) = 0$ is given by

$$x(t, s) = -\int_{\sigma(s)}^{t} e_p(\tau, \sigma(s))\Delta\tau. \tag{3.3}$$

Theorem 3.1. *Assume $Lx(t) = 0$ with (3.2) has only a trivial solution. For each fixed $s \in [a, b]$, let $u(t, s)$ be the unique solution of the BVP $Lu(t, s) = 0$, $u(a, s) = 0$, $u(\sigma^2(b), s) = -x(\sigma^2(b), s)$, where $x(t, s)$ is the Cauchy function for $Lx = 0$. Then*

$$G(t, s) = \begin{cases} u(t, s), & t \le s \\ u(t, s) + x(t, s), & \sigma(s) \le t \end{cases}$$

is Green's function for the BVP $Lx(t) = 0$ with (3.2). We can prove it similarly as that in [16].