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Integral manifolds of impulsive fractional functional differential systems

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ABSTRACT

In this paper, a class of impulsive Caputo fractional functional differential systems with variable impulsive perturbations is investigated. Sufficient conditions for the existence of integral manifolds are obtained. The main results are proved by means of piecewise continuous Lyapunov functions and the fractional comparison principle. The demonstrated techniques can be applied in studying properties of many applied problems of diverse interest.

1. Introduction

Fractional differential equations have become an active research subject in nonlinear science and have attracted more attention in many fields [1–3]. In relation to the mathematical simulation in chaos, fluid dynamics and many physical systems, recently the investigation of impulsive fractional differential equations began [4–6]. Besides impulsive effects, delay effects exist widely in many real-world models [7,8]. In recent years, the theory of impulsive functional differential equations has also been significantly developed; see [9–12] and the references therein. However, in spite of the great possibilities of applications, the theory of the impulsive fractional functional differential equations is still in the initial stage. There have appeared a few results for such equations with fixed moments of impulse effect [13–16] and variable moments are rarely considered in the literature. In the investigation of the impulsive differential equations with variable impulsive perturbations there arises a number of difficulties related to the phenomena of "beating" of the solutions, bifurcation, loss of the property of autonomy, etc [17]. To the best of our knowledge, the problem for the existence of integral manifolds for impulsive fractional functional differential equations with variable impulsive perturbations has not been studied previously.

Motivated by the above, we consider a class of impulsive fractional functional differential equations with variable impulsive perturbations. By means of piecewise continuous Lyapunov functions and the fractional comparison principle [15] sufficient conditions for the existence of integral manifolds are obtained.

2. Preliminaries

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with the norm $\|\cdot\|$, Ω be an open set in $\mathbb{R}^n$ containing the origin, and let $\mathbb{R}_+ = [0, \infty)$. For a given $r > 0$, we denote by $\mathcal{PC}$ the following space $\mathcal{PC} = PC[[-r, 0], \Omega] = \{\varphi : [-r, 0] \to \Omega : \varphi(t)$ is a piecewise continuous function with points of discontinuity $\tilde{t} \in [-r, 0]$ at which $\varphi(\tilde{t}^-)$ and $\varphi(\tilde{t}^+)$ exist and $\varphi(\tilde{t}^-) = \varphi(\tilde{t})\big\}$.

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Let $t_0 \in \mathbb{R}_+$. Consider the following system of impulsive fractional functional differential equations

$$\begin{cases} {}^cD^q x(t) = f(t, x_t), & t \neq \tau_k(x(t)), k = 1, 2, \ldots, \\ \Delta x(t) = x(t^+) - x(t^-) = I_k(x(t)), & t = \tau_k(x(t)), k = 1, 2, \ldots, \end{cases} \tag{2.1}$$

where $f : [t_0, \infty) \times \mathcal{PC} \to \mathbb{R}^n$, ${}^cD^q$ is Caputo's fractional derivative of order q, $0 < q < 1$, $I_k : \Omega \to \mathbb{R}^n$, $\tau_k : \Omega \to \mathbb{R}^n$, $k = 1, 2, \ldots$, and for any $t \geq t_0$, $x_t \in \mathcal{PC}$ is defined by $x_t(s) = x(t + s)$, $-r \leq s \leq 0$.

Definition 2.1 ([3,15]). For any $t \geq t_0$, *Caputo's fractional derivative of order* q, $0 < q < 1$ with the lower limit t_0 for a function $l \in C^1[[t_0, b], \mathbb{R}^n]$, $b > t_0$, is defined as

$$ {}^cD^q l(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} \frac{l'(s)}{(t-s)^q} ds.$$

Here and in what follows Γ denotes the Gamma function.

Let $\varphi_0 \in \mathcal{PC}$. Denote by $x(t) = x(t; t_0, \varphi_0)$ the solution of system (2.1) that satisfies the initial conditions:

$$\begin{cases} x(t; t_0, \varphi_0) = \varphi_0(t - t_0), & t_0 - r \leq t \leq t_0, \\ x(t_0^+; t_0, \varphi_0) = \varphi_0(0). \end{cases} \tag{2.2}$$

The solutions $x(t)$ of system (2.1) are piecewise continuous functions with points of discontinuity of the first kind in which they are left continuous; i.e., at the moments t_{l_k} when the integral curve of the solution $x(t)$ meets the hypersurfaces

$$\sigma_k = \Big\{(t, x) \in [t_0, \infty) \times \Omega : t = \tau_k(x)\Big\}$$

the following relations are satisfied:

$$x(t_{l_k}^-) = x(t_{l_k}), \qquad x(t_{l_k}^+) = x(t_{l_k}) + I_{l_k}(x(t_{l_k})).$$

The points $t_{l_1}, t_{l_2}, \ldots$ $(t_0 < t_{l_1} < t_{l_2})$ are the impulsive moments. Let us note that, in general, $k \neq l_k$. In other words, it is possible that the integral curve of the problem under consideration does not meet the hypersurface σ_k at the moment t_k. It is clear that the solutions of systems with variable impulsive perturbations have points of discontinuity depending on the solutions, i.e. the different solutions have different points of discontinuity. This leads to a number of difficulties in the investigation of such systems. One of the phenomena occurring with systems of type (2.1) is the so-called beating of the solutions. This is the phenomenon when the mapping point $(t, x(t))$ meets one and the same hypersurface σ_k several or infinitely many times [17].

Let $\tau_0(x) \equiv t_0$ for $x \in \Omega$. We assume that the functions $\tau_k(x)$ are continuous and the following relations hold:

$$t_0 < \tau_1(x) < \tau_2(x) < \cdots, \tau_k(x) \to \infty \quad \text{as } k \to \infty$$

uniformly on $x \in \Omega$. We also suppose that the functions f, I_k and τ_k are smooth enough on $[t_0, \infty) \times \Omega$ and Ω, respectively, to guarantee existence, uniqueness and continuability of the solution $x(t) = x(t; t_0, \varphi_0)$ of the initial value problem (IVP) (2.1), (2.2) on the interval $[t_0, \infty)$ for each $\varphi_0 \in \mathcal{PC}$, and $t_0 \in \mathbb{R}_+$ and absence of the phenomenon "beating".

For more results about such systems, we refer the reader to [11,17].

We shall introduce the following definition of integral manifolds connected with system (2.1).

Definition 2.2. We call an arbitrary manifold M in the extended phase space $[t_0 - r, \infty) \times \Omega$ of (2.1) *integral manifold*, if $(t, \varphi_0(t - t_0)) \in M$ for $t \in [t_0 - r, t_0]$ implies $(t, x(t)) \in M$, $t \geq t_0$.

Let $G_k = \Big\{(t, x) \in [t_0, \infty) \times \Omega : \tau_{k-1}(x) < t < \tau_k(x)\Big\}$, $k = 1, 2, \ldots$ and $G = \cup_{k=1}^{\infty} G_k$.

In the sequel, we shall use piecewise continuous auxiliary functions, which belong to the class V_M such that:

$V_M = \{V : [t_0, \infty) \times \Omega \to \mathbb{R}_+ : V \in C[G, \mathbb{R}_+], t \in [t_0, \infty),$ V is locally Lipschitz continuous with respect to its second argument on each of the sets G_k; $V(t, x) = 0$ for $(t, x) \in M$, $t \geq t_0$ and $V(t, x) > 0$ for $(t, x) \in [t_0, \infty) \times \Omega \setminus M$; for $(t_0^*, x_0^*) \in \sigma_k$, $V(t_0^{*-}, x_0^*)$ and $V(t_0^{*+}, x_0^*)$ exist, and $V(t_0^{*-}, x_0^*) = V(t_0^*, x_0^*)\}$.

Definition 2.3. Given a function $V \in V_M$. For $t \in G_k$, $k = 1, 2, \ldots$ and $\varphi \in \mathcal{PC}$ the *upper right-hand derivative of* V *in Caputo's sense of order* q, $0 < q < 1$ *with respect to system* (2.1) is defined by

$$ {}^cD_+^q V(t, \varphi(0)) = \lim_{h \to 0^+} \sup \frac{1}{h^q} [V(t, \varphi(0)) - V(t - h, \varphi(0) - h^q f(t, \varphi))].$$

Let $t_1, t_2, \ldots$ $(t_0 < t_1 < t_2 < \cdots)$ be the moments in which the integral curve $(t, x(t; t_0, \varphi_0))$ of problem (2.1), (2.2) meets the hypersurfaces σ_k, $k = 1, 2, \ldots$, i.e. each of the points t_k is a solution of some of the equations $t = \tau_k(x(t))$, $k = 1, 2, \ldots$.

In the proof of the main results we shall use the following results from [15].

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