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A new multivariate spline based on mixed partial derivatives and its finite element approximation

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ABSTRACT

We present a new multivariate spline using mixed partial derivatives. We show the existence and uniqueness of the proposed multivariate spline problem, and propose a simple finite element approximation.

1. Introduction

Multivariate splines are often used to interpolate and smooth scattered data [1,2]. A multivariate spline is given as follows. Let $\Omega \subset \mathbb{R}^d$ with $d \in \mathbb{N}$ be a closed and bounded region. Given a set $\mathcal{G} = \{\mathbf{p}_i\}_{i=0}^N$ of scattered points in $\bar{\Omega}$ and a set $\{z_i\}_{i=0}^N$, the multivariate L-spline is a smooth function $u : \Omega \to \mathbb{R}$ which satisfies

$$\underset{u \in V}{\operatorname{argmin}} \left(\sum_{i=1}^{N} (u(\mathbf{p}_i) - z_i)^2 + \lambda \int_{\Omega} (L_p u)^2 \, d\mathbf{x} \right), \tag{1.1}$$

where V is a Sobolev space, L_p is a linear partial differential operator, and λ is a positive constant. There are a few choices for L_p. The first choice is to take $L_p u$ as the Hessian of the smoother u leading to a thin plate spline [1,3]. Then we need to have $V = H^2(\Omega)$. There are two drawbacks of this approach. The first drawback is that a finite element approximation of this spline requires a H^2-conforming finite element space. One can use a mixed finite element space to obtain an efficient numerical technique as in [4] but the numerical scheme is still complicated. The other drawback is that the problem is not well-posed if the dimension $d > 3$. It is often very important to deal with a high-dimensional problem. The second choice is to choose $L_p u$ as the Laplacian of the smoother u. This choice also has the same two drawbacks as above.

The third choice, which is more practical for a high-dimensional problem, is to choose $L_p u$ as the gradient of the smoother u. In that case we have $V = H^1(\Omega)$. While we can apply a very efficient finite element scheme to approximate the solution in this case, the continuous problem is not well-posed with this choice. The obvious reason for this is the point value of a func-

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tion in $H^1(\Omega)$ is not defined when $d \geq 2$. This approach is very popular for a high-dimensional problem due to its efficiency although the ill-posedness of the problem in the continuous setting exhibits in the discrete setting for a fine mesh [5].

We aim at proposing a multivariate spline problem which is suitable for a high-dimensional data problem, and which allows an efficient finite element approximation. There are some finite element approaches for the smoothing problem [2,4,6–8]. Some of them [2,4,8] are direct finite element approximations of the continuous problem proposed in [1,3], whereas a new finite element spline is proposed in [7]. A finite element method [7,9] is more efficient for a high-dimensional problem with a large data set than a traditional radial basis function approach [10] since the arising linear system is sparse and does not depend on the number of data points.

In this paper we introduce a very simple multivariate spline using a mixed partial derivative of the smoother. This new multivariate spline is defined for any dimension $d \in \mathbb{N}$, and therefore, very useful for a high-dimensional data smoothing problem. This multivariate spline also allows an efficient finite element approximation. Therefore, it can be applied to a problem with a large data set. Since a finite element method can be efficiently used to approximate the solution of a variational problem, we propose a variational formulation of the multivariate problem.

This paper is organized as follows. We present our multivariate spline in the next section. We show the existence of a unique solution of the multivariate spline problem. A finite element method is outlined in Section 3. Finally, a conclusion is drawn in the last section.

2. A new multivariate spline

Let $\mathcal{B} = \{0, 1\}^d \setminus \{\mathbf{0}\}$, where $\mathbf{0} \in \mathbb{R}^d$ is a zero vector. We consider a standard multi-index notation with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathcal{B}$ so that a mixed derivative of a sufficiently smooth function u is denoted by

$$D^{\boldsymbol{\alpha}} u = \frac{\partial^{\sum_{i=1}^d \alpha_i}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} u,$$

where we use the usual Cartesian coordinate system with $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. We use the standard notation for Sobolev spaces on Ω [11–13]. The set of all square-integrable functions in Ω is denoted by $L^2(\Omega)$; and

$$H^1(\Omega) := \left\{u \in L^2(\Omega) : \nabla u \in [L^2(\Omega)]^d\right\}, \quad \text{and}$$
$$H^1_m(\Omega) := \left\{u \in L^2(\Omega) : D^{\boldsymbol{\alpha}} u \in L^2(\Omega),\ \boldsymbol{\alpha} \in \mathcal{B}\right\},$$

where these spaces are equipped with norms

$$\|u\|_{L^2(\Omega)} = \sqrt{\int_\Omega u^2\, d\mathbf{x}}, \qquad \|u\|_{H^1(\Omega)} = \sqrt{\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)}}, \quad \text{and}$$
$$\|u\|_{H^1_m(\Omega)} = \sqrt{\|u\|^2_{L^2(\Omega)} + \sum_{\boldsymbol{\alpha} \in \mathcal{B}} \|D^{\boldsymbol{\alpha}} u\|^2_{L^2(\Omega)}}, \quad \text{respectively.}$$

Note that semi-norms on $H^1(\Omega)$ and $H^1_m(\Omega)$ are defined as

$$|u|_{H^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}, \quad \text{and} \quad |u|_{H^1_m(\Omega)} = \sqrt{\sum_{\boldsymbol{\alpha} \in \mathcal{B}} \|D^{\boldsymbol{\alpha}} u\|^2_{L^2(\Omega)}},$$

respectively. We note that the space $H^1_m(\Omega)$ is a Hilbert space, and $H^1_m(\Omega) \subset C^0(\Omega)$ [14].

Our new multivariate spline is a smooth function $u : \Omega \to \mathbb{R}$ which is the minimum of

$$\underset{u \in V}{\operatorname{argmin}} \left[\sum_{i=1}^N (u(\mathbf{p}_i) - z_i)^2 + \lambda \left(\sum_{\boldsymbol{\alpha} \in \mathcal{B}} \|D^{\boldsymbol{\alpha}} u\|^2_{L^2(\Omega)}\right)\right], \tag{2.1}$$

where V is the Sobolev space $H^1_m(\Omega)$. Since $H^1_m(\Omega) \subset C^0(\Omega)$, the function values $\{u(\mathbf{p}_i)\}_{i=1}^N$ are well-defined. In the following we assume that $V = H^1_m(\Omega)$.

We now show that the multivariate spline problem is well-posed. In order to show this we introduce a bilinear form $a(\cdot, \cdot)$ defined by

$$a(u, v) = (Pu)^T P v + \lambda \left(\sum_{\boldsymbol{\alpha} \in \mathcal{B}} \int_\Omega D^{\boldsymbol{\alpha}} u D^{\boldsymbol{\alpha}} v\, d\mathbf{x}\right),$$

and a linear form $\ell(\cdot)$ defined by

$$\ell(v) = (Pv)^T \mathbf{z},$$

where

$$Pu = (u(\mathbf{p}_0), u(\mathbf{p}_1), \ldots, u(\mathbf{p}_N))^T,$$