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An efficient family of weighted-Newton methods with optimal eighth order convergence

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ABSTRACT

Based on Newton's method, we present a family of three-point iterative methods for solving nonlinear equations. In terms of computational cost, the family requires four function evaluations and has convergence order eight. Therefore, it is optimal in the sense of Kung–Traub hypothesis and has the efficiency index 1.682 which is better than that of Newton's and many other higher order methods. Some numerical examples are considered to check the performance and to verify the theoretical results. Computational results confirm the efficient and robust character of presented algorithms.

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1. Introduction

In this study, we consider iterative methods for solving the nonlinear equation $f(x) = 0$, where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D. Newton's method is probably the most widely used algorithm for solving such equations, which starts with an initial approximation x_0 closer to a root (say, r) and generates a sequence of successive iterates $\{x_i\}_{i=0}^{\infty}$ converging quadratically to the root r. It is given by

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \tag{1}$$

where $f'(x)$ is the first order derivative of the function $f(x)$. In order to improve the local order of convergence of Newton's method a number of modified methods have been developed in the literature, see for example [1–27] and references therein.

Ostrowski [1] proposed the concept of efficiency index as a measure for comparing the efficiency of different methods. This index is described by $E = p^{1/m}$, wherein p is the order of convergence and m is the total number of function evaluations needed per iteration. Later on, Kung and Traub [28] conjectured that multipoint methods without memory [29] based on m function evaluations have the optimal order of convergence 2^{m-1}. Multipoint methods with this property are usually called optimal methods. For example, with three function evaluations a two-point method of optimal fourth order convergence can be constructed (see [1–8]) and with four function evaluations a three-point method of optimal eighth order convergence can be developed (see [9–20]). A more extensive list of references as well as a survey on progress made on the class of multipoint methods may be found in the recent book by Petković et al. [30].

In this paper, our aim is to develop an iterative method that may satisfy the basic requirements of generating a quality numerical algorithm, that is, an algorithm which has (i) high convergence speed, (ii) minimum computational cost, and (iii) simple structure. Thus, we derive a new family of three-point methods with optimal eighth order of convergence. The

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scheme is composed of three steps of which first two steps consist of any fourth order method with the base as well-known Newton's iteration and the third step is weighted Newton iteration. Rest of the paper is organized as follows. In Section 2 the new family is developed and its convergence analysis is discussed. The theoretical results proved in Section 2 are verified in Section 3 through numerical experimentation along with a comparison of the new methods with the existing methods of same class.

2. The methods and analysis of convergence

Based on the above considerations of a quality numerical algorithm, we begin with the three-point iteration scheme

$$\begin{cases} y_i = x_i - \dfrac{f(x_i)}{f'(x_i)}, \\ z_i = M_4(x_i, y_i), \\ x_{i+1} = z_i - \dfrac{f'(x_i) - f[y_i, x_i] + f[z_i, y_i]}{2f[z_i, y_i] - f[z_i, x_i]} \dfrac{f(z_i)}{f'(x_i)}. \end{cases} \tag{2}$$

Here $M_4(x_i, y_i)$ is any optimal fourth order scheme with the base as Newton's iteration y_i and $f[\cdot,\cdot]$ is Newton's first order divided difference. Through the following theorem we prove that this scheme has eighth order of convergence.

Theorem 1. *Let the function $f(x)$ be sufficiently differentiable in a neighborhood of its zero r and $M_4(x_i, y_i)$ is an optimal fourth order method which satisfies*

$$z_i - r = B_0 e_i^4 + B_1 e_i^5 + B_2 e_i^6 + B_3 e_i^7 + B_4 e_i^8 + O(e_i^9), \tag{3}$$

where $B_0 \neq 0$ and $e_i = x_i - r$. If an initial approximation x_0 is sufficiently close to r, then the order of convergence of (2) is at least 8.

Proof. Let $\tilde{e}_i = y_i - r$ and $\hat{e}_i = z_i - r$ be the errors in the i-th iteration. Using Taylor's expansion of $f(x_i)$ about r and taking into account that $f(r) = 0$ and $f'(r) \neq 0$, we have

$$f(x_i) = f'(r)[e_i + A_2 e_i^2 + A_3 e_i^3 + \cdots + A_8 e_i^8 + O(e_i^9)], \tag{4}$$

where $A_k = (1/k!) f^{(k)}(r)/f'(r),\ k = 2, 3, 4, \ldots$.
Also,

$$f'(x_i) = f'(r)[1 + 2A_2 e_i + 3A_3 e_i^2 + \cdots + 9A_9 e_i^8 + O(e_i^9)]. \tag{5}$$

Substitution of (4) and (5) in the first step of (2) gives

$$\begin{aligned} \tilde{e}_i = {} & A_2 e_i^2 + (-2A_2^2 + 2A_3) e_i^3 + (4A_2^3 - 7A_2A_3 + 3A_4) e_i^4 - 2(4A_2^4 - 10A_2^2A_3 + 3A_3^2 + 5A_2A_4 - 2A_5) e_i^5 \\ & + (16A_2^5 - 52A_2^3A_3 + 33A_2A_3^2 + 28A_2^2A_4 - 17A_3A_4 - 13A_2A_5 + 5A_6) e_i^6 \\ & - 2(16A_2^6 - 64A_2^4A_3 + 63A_2^2A_3^2 - 9A_3^3 + 36A_2^3A_4 - 46A_2A_3A_4 + 6A_4^2 - 18A_2^2A_5 + 11A_3A_5 \\ & + 8A_2A_6 - 3A_7) e_i^7 + (64A_2^7 - 304A_2^5A_3 + 408A_2^3A_3^2 - 135A_2A_3^3 + 176A_2^4A_4 - 348A_2^2A_3A_4 + 75A_3^2A_4 \\ & + 64A_2A_4^2 - 92A_2^3A_5 + 118A_2A_3A_5 - 31A_4A_5 + 44A_2^2A_6 - 27A_3A_6 - 19A_2A_7 + 7A_8) e_i^8 + O(e_i^9). \end{aligned} \tag{6}$$

Expanding $f(y_i)$ about r, we obtain

$$f(y_i) = f'(r)[\tilde{e}_i + A_2 \tilde{e}_i^2 + A_3 \tilde{e}_i^3 + A_4 \tilde{e}_i^4 + O(\tilde{e}_i^5)]. \tag{7}$$

Also, expansion of $f(z_i)$ about r yields

$$f(z_i) = f'(r)[\hat{e}_i + A_2 \hat{e}_i^2 + O(\hat{e}_i^3)]. \tag{8}$$

Substituting Eqs. (3)–(8) in the third step of (2) and simplifying, we obtain

$$e_{i+1} = x_{i+1} - r = -B_0(A_3^2 + A_2(-A_4 + B_0)) e_i^8 + O(e_i^9). \tag{9}$$

This completes the proof of Theorem 1. □

Thus, the scheme (2) defines a new family of three-point eighth order methods which utilizes four function evaluations, namely $f(x_i)$, $f(y_i)$, $f(z_i)$ and $f'(x_i)$. This family is, therefore, optimal in the sense of Kung–Traub conjecture. The efficiency index E for this family is, $8^{1/4} \approx 1.682$ which is better than the efficiency of Newton's method, fourth order methods [1–8], sixth order methods [21–24] and seventh order methods [25–27] whose E-values are: $2^{1/2} \approx 1.414$, $4^{1/3} \approx 1.587$, $6^{1/4} \approx 1.565$ and $7^{1/4} \approx 1.627$, respectively. However, this E-value is equal to the E-values of existing eighth order methods cited in the previous section. Some simple members of the family (2) are as follows:

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