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Achieving synchronization between the fractional-order hyperchaotic Novel and Chen systems via a new nonlinear control technique

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ABSTRACT

In this work, we discuss the stability conditions for a nonlinear fractional-order hyperchaotic system. The fractional-order hyperchaotic Novel and Chen systems are introduced. The existence and uniqueness of solutions for two classes of fractional-order hyperchaotic Novel and Chen systems are investigated. On the basis of the stability conditions for nonlinear fractional-order hyperchaotic systems, we study synchronization between the proposed systems by using a new nonlinear control technique. The states of the fractional-order hyperchaotic Novel system are used to control the states of the fractional-order hyperchaotic Chen system. Numerical simulations are used to show the effectiveness of the proposed synchronization scheme.

1. Introduction

In recent years, differential equations with fractional order have attracted many researchers because of their useful applications in many fields such as physics [1], engineering [2,3], mathematical biology [4,5], finance [6] and social sciences [7]. Meanwhile, chaos is one of the most fascinating phenomena to have been extensively studied and developed by scientists. Chaos has also useful applications like chaos synchronization, which has attracted particular interest in the past few years [8]. Recently, chaos synchronization has been shown to have potential applications in chemical reactors [9], biological networks [10], artificial neural networks [11] and secure communications [12].

Lyapunov exponents are measures that quantify the chaotic behaviors. A regular chaotic system has one positive Lyapunov exponent. However, a system with more than one positive Lyapunov exponent is called "hyperchaotic". Thus, the hyperchaotic system has more complex behaviors and abundant dynamics than the chaotic system, so the hyperchaotic system has more applications than the chaotic system especially in secure communications [13]. Thus, fractional-order hyperchaotic systems are more promising for applications in secure communications. Recently, some fractional-order hyperchaotic systems have been investigated, such as the fractional-order hyperchaotic Rössler system [14], the fractional-order hyperchaotic Chen system [15] and the fractional-order hyperchaotic Novel system [16].

In this Letter, we introduce a new nonlinear control scheme for achieving synchronization between the fractional-order hyperchaotic Novel system and the fractional-order hyperchaotic Chen system; the fractional-order hyperchaotic Novel system is used to drive the fractional-order hyperchaotic Chen system. The proposed hyperchaotic systems and technique have possible applications in modeling fractional-order hyperchaotic circuits; see for example Refs. [17,18].

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2. Fractional calculus

One of the most common definitions of fractional derivatives is the Caputo definition [19]:

$$D^{\alpha}f(t) = I^{m-\alpha}f^{(m)}(t), \quad \alpha > 0, \tag{1}$$

where m is the least integer which is not less than α, and I^{β} is the Riemann–Liouville integral operator of order β which is described as follows:

$$I^{\beta}g(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}g(\tau)d\tau, \quad \beta > 0, \tag{2}$$

where $\Gamma(\beta)$ is Euler's Gamma function. The operator D^{α} is generally called the "Caputo differential operator of order α". Consider the initial value problem

$$D^{\alpha}\mathrm{X}(t) = f(t, \mathrm{X}(t)), \quad 0 \le t \le T,\ \mathrm{X}^{(k)}(0) = \mathrm{X}_0^{(k)},\ k = 0, 1, \ldots, m-1. \tag{3}$$

Theorem 1 (*Existence [20]*). *Assume that* $\mathrm{E} := [0, \chi^*] \times [\mathrm{X}_0^{(0)} - \varepsilon, \mathrm{X}_0^{(0)} + \varepsilon]$ *with some* $\chi^* > 0$ *and some* $\varepsilon > 0$, *and let the function* $f : \mathrm{E} \to R$ *be continuous. In addition, let* $\chi := \min\{\chi^*, (\varepsilon\Gamma(\alpha+1)/\|f\|_{\infty})^{1/\alpha}\}$. *Then, there exists a function* $\mathrm{X} : [0, \chi] \to R$ *solving the initial value problem (3).*

Theorem 2 (*Uniqueness [20]*). *Suppose that* $\mathrm{E} := [0, \chi^*] \times [\mathrm{X}_0^{(0)} - \varepsilon, \mathrm{X}_0^{(0)} + \varepsilon]$ *with some* $\chi^* > 0$ *and some* $\varepsilon > 0$. *Moreover, let the function* $f : \mathrm{E} \to R$ *be bounded on* E *and satisfy a Lipschitz condition with respect to the second variable, i.e.* $|f(t, \mathrm{X}) - f(t, \Upsilon)| \le L|\mathrm{X} - \Upsilon|$ *with some constant* $L > 0$ *independent of* t, X *and* Υ. *Then, defining* χ *as in Theorem 1, there exists at most one function* $\mathrm{X} : [0, \chi] \to R$ *solving the initial value problem (3).*

The local stability of the equilibrium points of a linear fractional-order system is governed by the following results of Matignon [21]:

$$|\arg(\lambda_i)| > \alpha\pi/2, \quad (i = 1, 2, 3, 4), \tag{4}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of the equilibrium points. Now, consider the following nonlinear fractional-order hyperchaotic system:

$$D^{\alpha}\mathrm{X}(t) = f(X(t)), \qquad \mathrm{X} = \mathrm{X}_0 \tag{5}$$

where $\mathrm{X}(t) = (x_1, x_2, x_3, x_4)^T \in R^4$, $f : R^4 \to R^4$ is a nonlinear vector function in terms of X. For a small perturbation δ around the equilibrium point X^*, system (5) can be given as

$$D^{\alpha}\mathrm{X}(t) = J(\mathrm{X}^*)\mathrm{X} + g(\mathrm{X}^*),$$

where $J(\mathrm{X}^*) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij}\Big|_{\mathrm{X}=\mathrm{X}^*}$ is the Jacobian matrix evaluated at the equilibrium point $\mathrm{X}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$, and $g(\mathrm{X}^*)$ is a continuous nonlinear function. Hence, we have the following lemma:

Lemma 1. *The equilibrium point* $\mathrm{X}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$ *of the nonlinear system (5) is locally asymptotically stable if all the eigenvalues* $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *of the Jacobian matrix* J *satisfy the conditions (4).*

Proof. Consider the four-dimensional nonlinear autonomous fractional-order hyperchaotic system

$$\begin{aligned} &D^{\alpha}x_1(t) = f_1(x_1, x_2, x_3, x_4), \qquad D^{\alpha}x_2(t) = f_2(x_1, x_2, x_3, x_4), \qquad D^{\alpha}x_3(t) = f_3(x_1, x_2, x_3, x_4), \\ &D^{\alpha}x_4(t) = f_4(x_1, x_2, x_3, x_4). \end{aligned} \tag{6}$$

Let $x_i(0) = x_{i0}$ be the initial values of system (6); then putting $x_i(t) = x_i^* + \delta_i(t)$, we get

$$D^{\alpha}(x_i^* + \delta_i) = f_i(x_1^* + \delta_1, x_2^* + \delta_2, x_3^* + \delta_3, x_4^* + \delta_4), \quad i = 1, 2, 3, 4$$

which leads to $D^{\alpha}\delta_i(t) = f_i(x_1^*+\delta_1, x_2^*+\delta_2, x_3^*+\delta_3, x_4^*+\delta_4)$. Using the Taylor expansion and the fact that $f_i(x_1^*, x_2^*, x_3^*, x_4^*) = 0$, we have

$$D^{\alpha}\delta_i(t) \approx \frac{\partial f_i}{\partial x_j}\Big|_{x=x^*}\delta_j, \quad j = 1, 2, 3, 4$$

which reduces to the following system:

$$D^{\alpha}\delta = J\delta, \quad \delta = (\delta_1, \delta_2, \delta_3, \delta_4)^T, \tag{7}$$

where $J(\mathrm{X}^*)$ satisfies the relation $B^{-1}JB = C$, $C = diag(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, and B is the eigenvector of J. System (7) has the initial values

$$\delta_1(0) = x_1(0) - x_1^*, \qquad \delta_2(0) = x_2(0) - x_2^*, \qquad \delta_3(0) = x_3(0) - x_3^*, \qquad \delta_4(0) = x_4(0) - x_4^*.$$