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A new existence result of positive solutions for the Sturm–Liouville boundary value problems☆

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ABSTRACT

We prove a new existence result of positive solutions for the Sturm–Liouville problems. Different from the existing research, we do not assume that a nonlinearity term f has numerical and functional lower bounds or satisfies some usual limit conditions.

1. Introduction

In 1996, using the fixed point theorems in a cone, Anuradha et al. [1] proved the existence of positive solutions for the following Sturm–Liouville problems under the assumptions of $g(t) = \lambda > 0$ and the semi-positone condition (that is, $f(t, z) \geq -h$ on $[0, 1] \times [0, \infty)$, where $h > 0$ is constant):

$$(p(t)z'(t))' + g(t)f(t, z(t)) = 0 \quad \text{on } (0, 1) \tag{1.1}$$

subject to the boundary conditions

$$\alpha z(0) - \beta p(0)z'(0) = 0, \qquad \gamma z(1) + \delta p(1)z'(1) = 0, \tag{1.2}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\Gamma := \gamma\beta + \alpha\gamma \int_0^1 \frac{1}{p(\mu)} d\mu + \alpha\delta > 0$. Sun and Zhang [2] investigated the existence of positive solutions of singular and sublinear cases for (1.1)–(1.2) under some conditions on f concerning the first eigenvalue corresponding to the relevant linear operator; they assumed $p \in C^1[0, 1]$ (specially, if f has no lower bound, then $f(0) = 0$, $q(x) < 0$ on $[0, 1]$ must be satisfied, see Theorem 3 [2]). In 2010, Yao [3] improved the semi-positone condition to a non-numerical lower bound, *i.e.*, there exists a function $h \in C(0, 1) \cap L[0, 1]$ with $h(t) \geq 0$ *a.e.* $[0, 1]$ such that $f(t, z) \geq -h(t)$ *a.e.* $[0, 1] \times [0, \infty)$. If f satisfies some usual limit conditions such as $f_\infty = \lim_{z\to\infty} \inf_{t\in[0,1]} \frac{f(t,z)}{z} > -\infty$, one may refer to [4–6] and the references therein for the existence of positive solutions of (1.1)–(1.2). Employing lower and upper solutions, variational methods and the global bifurcation theory of P. H. Rabinowitz, Cui, Sun and Zou [7], Tian and Ge [8], Benmezaï [9] studied the existence of multiple solutions and sign-changing solutions of (1.1)–(1.2), respectively, where $f_\infty > -\infty$ [7] or $f(t, z)$ is the Lipschitz continuous for z uniformly and $f_t'(t, z)$ exists [8] or f is a continuous function which is $o(|z|)$ near 0, $\lim_{z\to\infty} f'(z)$ and $\lim_{z\to-\infty} f'(z)$ exist and are finite [9].

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The present paper is motivated by [2,5,6], we prove a new existence result of positive solutions for a class of the Sturm–Liouville problems when f do not satisfy all assumptions mentioned above (see, Example 3.1). Our methods are first to investigate the property of nonzero solutions of an integral equation and then utilize the Leray–Schauder fixed point theorem in the Banach space to obtain the existence of positive solutions for (1.1)–(1.2).

2. Some preliminaries

In this paper, we assume that $f : [0, 1]\times\mathbb{R}^+ \to (-\infty, \infty)$ and $p : [0, 1] \to \mathbb{R}^+\setminus\{0\}$ are continuous, $g \in L[0, 1]\cap C(0, 1)$, $g(t) \geq 0$ on $(0, 1)$, g may be singular at $t = 0, 1$ and $\int_0^1 g(s)ds > 0$, where $\mathbb{R}^+ = [0, \infty)$.

A function z is called a positive solution of (1.1)–(1.2) if $z \in C^1[0, 1]\cap C^2(0, 1)$ with $z(t) > 0$ on $(0, 1)$ satisfies (1.1)–(1.2).

Let $C[0, 1]$ be a continuous function space with norm $\|z\| = \max\{|z(t)| : t \in [0, 1]\}$. It is well known that z is a positive solution of (1.1)–(1.2) if and only if $z \in C[0, 1]$ with $z(t) > 0$ on $(0, 1)$ satisfies the following integral equation [1,3,5]:

$$z(t) = \int_0^1 G(t, s)g(s)f(s, z(s))\,ds := Az(t) \quad \text{for } t \in [0, 1], \tag{2.1}$$

where $G(t, s)$ is Green's function to $-(p(t)z'(t))' = 0$ associated with the boundary conditions (1.2) defined by

$$G(t, s) = \frac{1}{\Gamma}\begin{cases} \left(\delta + \gamma\int_t^1 \frac{1}{p(\mu)}d\mu\right)\left(\beta + \alpha\int_0^s \frac{1}{p(\mu)}d\mu\right), & s \leq t,\\ \left(\beta + \alpha\int_0^t \frac{1}{p(\mu)}d\mu\right)\left(\delta + \gamma\int_s^1 \frac{1}{p(\mu)}d\mu\right), & t < s,\end{cases} \tag{2.2}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and Γ is in (1.2).

Letting $z \in C[0, 1]$ and $z^+(t) = \max\{z(t), 0\}$, we define a map A^* from $C[0, 1]$ to $C[0, 1]$ by

$$A^*z(t) = \int_0^1 G(t, s)g(s)f(s, z^+(s))\,ds.$$

The following theorem plays a key role in the study of the existence of positive solutions (1.1)–(1.2).

Theorem 2.1. *Suppose there exists a constant $r_0 > 0$ such that*

$$(P_f) \qquad f(t, y) \geq 0, \quad 0 \leq t \leq 1, 0 \leq y \leq r_0.$$

If $z = \lambda A^*z$ has a solution $z \in C[0, 1] \setminus \{0\}$ for some $\lambda > 0$, then $z(t) > 0$ for $t \in (0, 1)$.

Proof. (1) We first show that z has the following property.

Let $[a, b] \subseteq [0, 1]$, $z(t) \leq r_0$ for $t \in [a, b]$; if $\nu \in (a, b)$ such that $z'(\nu) = 0$, then $z(\nu) \geq \max\{z(a), z(b)\}$.

Differentiating z with t twice, we have

$$(p(t)z'(t))' = -\lambda g(t)f(t, z^+(t)) \quad \text{on } (0, 1). \tag{2.3}$$

By (P_f), $0 \leq z^+(t) \leq r_0$ for $t \in [a, b]$ and (2.3), then

$$(p(t)z'(t))' = -\lambda g(t)f(t, z^+(t)) \leq 0 \quad \text{on } (a, b).$$

Integrating this inequality from t to ν and from ν to t, respectively, we have by $z'(\nu) = 0$

$$p(t)z'(t) = \lambda\int_t^\nu g(s)f(s, z^+(s))ds \geq 0 \quad \text{for } t \in [a, \nu],$$

$$p(t)z'(t) = -\lambda\int_\nu^t g(s)f(s, z^+(s))ds \leq 0 \quad \text{for } t \in [\nu, b].$$

Then by $p(t) > 0$, $z'(t) \geq 0$ for $t \in [a, \nu]$ and $z'(t) \leq 0$ for $t \in [\nu, b]$. This implies $z(\nu) \geq z(a)$ and $z(\nu) \geq z(b)$.

(2) We show that there exists $\widetilde{t} \in (0, 1)$ satisfying $z(\widetilde{t}) > 0$. In fact, if $z(t) \leq 0$ for $t \in (0, 1)$, then $z^+(t) = 0$ for $t \in (0, 1)$. By (P_f), we have $f(s, 0) \geq 0$ and

$$z(t) = \lambda\int_0^1 G(t, s)g(s)f(s, z^+(s))\,ds = \lambda\int_0^1 G(t, s)g(s)f(s, 0)\,ds \geq 0$$

for $t \in [0, 1]$ and then $z(t) = 0$ for $t \in [0, 1]$, which contradicts $z \neq 0$.

(3) $z(0) \geq 0$ and $z(1) \geq 0$.

In fact, if $z(0) < 0$, by $z(\widetilde{t}) > 0$, we may choose an interval $[0, b] \subseteq [0, \widetilde{t})$ such that

$$z(t) < 0, \quad t \in [0, b), \; z(b) = 0.$$