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Tamed-Euler method for hybrid stochastic differential equations with Markovian switching

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ABSTRACT

This paper develops a numerical scheme for approximating solutions of stochastic differential equations with Markovian switching under such conditions that allow drift coefficients being locally one-sided Lipschitz continuous, and diffusion coefficients being locally Lipschitz continuous. The strong convergence of the algorithm is proved. In addition, under the assumption of polynomial growth rate of drift and global Lipschitz continuity, the classical rate of convergence is also obtained. Some numerical examples are provided for demonstration purpose.

1. Introduction

This work focuses on numerical solutions for stochastic differential equations with regime switching. Our main effort is devoted to obtaining convergence and rates of convergence of the numerical solutions under general conditions that allow the drift and the diffusion coefficient being locally Lipschitz and the drift having polynomial growth.

Because of the needs in various real-world applications, regime-switching diffusions have drawn much attention in recent years. Regime-switching diffusions, also known as hybrid switching diffusions, have two components. One of them is the continuous state component as in the usual diffusions, and the other is a switching component represented by a pure jump process. In such systems, continuous dynamics and discrete events coexist. The models have been used in such applications as option pricing in finance [1], jump linear systems in automatic control [2], hierarchical decision making in production planning [3], differential games with switching [4], estimation in hybrid systems [5], stock liquidation [6], and competitive Lotka–Volterra models in random environments [7,8], among others.

Explicit solutions are almost impossible to obtain for such systems due to the nonlinearity and the random switching. Thus, numerical solutions become vitally important. In the last decade, many results on numerical methods have been obtained for stochastic differential equations with regime switching under various conditions. One of the first was the paper [9] on the Euler–Maruyama scheme. In [10], the L_1 and L_2 convergence of Euler–Maruyama method for stochastic differential equations with Markovian switching was obtained under certain non-Lipschitz conditions. Strong pathwise convergence for weak Euler–Maruyama method for diffusions with Markovian switching was treated in [11]. Milstein-type method was studied in [12]. Weak convergence and strong convergence of the Euler–Maruyama methods for jump diffusions

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with Markovian switching were both considered in [13]. Rates of convergence to invariant measures were obtained in [14]. Most of the work mentioned above assumed the linear growth and Lipschitz conditions for the drift and diffusion coefficients. Nevertheless, in many applications, these assumptions are violated. As an example, we mention the following Lotka–Volterra system.

Example 1.1. In environment modeling, the stochastic Lotka–Volterra ecosystem in a random environment is given by

$$dx_i(t) = x_i(t)\Big\{\Big[r_i(\alpha(t)) - \sum_{j=1}^{n} a_{ij}(\alpha(t))x_j(t)\Big]dt + \sigma_i(\alpha(t))dB_i(t)\Big\}, \quad i = 1, 2, \ldots, d, \tag{1}$$

where $\alpha(\cdot)$ is a Markov chain taking values in a finite state space $\mathcal{M}$, $B(\cdot) = (B_1(\cdot), \ldots, B_d(\cdot))^\top$ is a d-dimensional standard Brownian motion, and $r_i(\cdot)$, $a_{ij}(\cdot)$ and $\sigma_i(\cdot)$ are functions defined on $\mathcal{M}$ for $i, j = 1, \ldots, d$. Without loss of generality, we assume that the initial conditions $x(0)$ and $\alpha(0)$ are non-random and that the Markov chain $\alpha(\cdot)$ and the Brownian motion $B(\cdot)$ are independent. For $i_0 \in \mathcal{M}$, denote $b_i(i_0) = r_i(i_0) - \frac{1}{2}\sigma_i^2(i_0)$ for $i = 1, \ldots, d$, $b(i_0) = (b_1(i_0), \ldots, b_d(i_0))^\top$, $A(i_0) = (a_{ij}(i_0))_{d\times d}$, and $\Sigma(i_0) = \operatorname{diag}(\sigma_1(i_0), \ldots, \sigma_d(i_0))$. Assume that $b_i(i_0) > 0$ for each $i_0 \in \mathcal{M}$ and $i = 1, \ldots, d$. Then $b(i_0)$, $A(i_0)$, and $\Sigma(i_0)$ represent different growth rates, community matrices, and noise intensities in different external environments, respectively. One can easily see that the drift coefficient in the system (1) does not satisfy the linear growth and the global Lipschitz conditions.

It is widely recognized that systems grow faster than linear may cause problems in numerical approximation. When we construct numerical algorithms, if we ignore the fast growth rate and naively put together a numerical algorithm, it may perform poorly or fail to converge. In fact, in the later section, we provide examples in which the standard Euler method fails to produce convergent iterates. Therefore, modifications are needed.

Inspired by the work [15] and [16], in this paper, we propose a tamed-Euler scheme for stochastic differential equations with Markov switching to deal with the case of local Lipschitz drift coefficient having polynomial growth rate. Roughly, a "tamed" scheme is one that properly modifies the coefficients of the systems so as to confine the iterates in a reasonable range. Thus a tamed algorithm can be viewed as one with "soft" constraints. The papers [15,16] concern the tamed-Euler algorithm for stochastic differential equations without Markov switching. In our present paper, with a Markov switching included, the original hybrid stochastic differential equations and the corresponding schemes become more complicated since the continuous dynamics and discrete events coexist and are intertwined.

The rest of the paper is organized as follows. The next section formulates the problem and gives the assumptions needed. Section 3 presents the numerical method and the convergence of the algorithm. Section 4 provides the proof of the auxiliary estimates and the main results. Section 5 gives the simulation results. Section 6 provides further remarks. Finally, for continuity of presentation, proofs of some technical lemmas are relegated in an Appendix at the end of the paper.

2. Formulation

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. On this probability space, let $B(t)$ be an $\mathbb{R}^d$-valued standard Brownian motion and $\{\alpha(t), t \ge 0\}$ a continuous-time Markov chain taking values in the finite set $\mathcal{M} = \{1, 2, \ldots, m_0\}$ with generator $Q = \big(q_{i_0 j_0}\big)_{i_0, j_0 \in \mathcal{M}}$ satisfying $q_{i_0 j_0} \ge 0$ for $i_0, j_0 \in \mathcal{M}$, $i_0 \ne j_0$ and $\sum_{j_0 \in \mathcal{M}} q_{i_0 j_0} = 0$ for each $i_0 \in \mathcal{M}$. For a fixed positive number T, let $\mathcal{B}([0, T])$ and $\mathcal{B}(\mathbb{R}^d)$ denote the Borel σ fields of $[0, T]$ and $\mathbb{R}^d$, respectively. For each vector $x \in \mathbb{R}^d$, let $x^\top$ and $|x|$ denote its transpose and its norm, respectively. Let X_0 be a $\mathbb{R}^d$-valued random variable satisfying $E|X_0|^q < \infty$ for some $q > 0$. For each $t \ge 0$, denote $\mathcal{F}_t = \sigma\big\{X_0, \alpha(s), B(s) : 0 \le s \le t\big\}$. Throughout the paper we assume that $B(\cdot)$, $\alpha(\cdot)$, and X_0 are independent. In addition, we use C and C_R to denote generic constants that may change from place to place.

Consider the following stochastic differential equation

$$dX(t) = b(t, X(t), \alpha(t))dt + \sigma(t, X(t), \alpha(t))dB(t), \quad X(0) = X_0, \ t \in [0, T], \tag{2}$$

where for each $i_0 \in \mathcal{M}$, $b(\cdot, \cdot, i_0) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma(\cdot, \cdot, i_0) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable vector-valued functions. We pose the following assumptions.

(A1) There exists a constant K such that

$$x^\top b(t, x, i_0) \vee |\sigma(t, x, i_0)|^2 \le K(1 + |x|^2)$$

for all $t \in [0, T]$, $x \in \mathbb{R}^d$, and $i_0 \in \mathcal{M}$.

(A2) For every $R > 0$, there exists a constant $K_R > 0$ such that

$$(x - y)^\top [b(t, x, i_0) - b(t, y, i_0)] \vee |\sigma(t, x, i_0) - \sigma(t, y, i_0)|^2 \le K_R |x - y|^2$$

for all $t \in [0, T]$, $|x|, |y| \le R$, and $i_0 \in \mathcal{M}$.

(A3) For every $R \ge 0$, there exists a constant $N_R > 0$ such that

$$\sup_{|x|\le R} |b(t, x, i_0)| \le N_R$$

for all $t \in [0, T]$ and $i_0 \in \mathcal{M}$.