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Almost sure exponential stability and stochastic stabilization of stochastic differential systems with impulsive effects

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ABSTRACT

This paper focuses on the problem of almost sure exponential stability and stochastic stabilization of nonlinear stochastic differential systems with impulsive effects. The moment stability analysis of impulsive stochastic differential systems has received considerable attention. But relatively little is known about the almost sure exponential stability and noise stabilization. In this paper, by using Lyapunov function, we shall not only establish the general criteria on almost sure exponential stability for general nonlinear impulsive stochastic differential systems but also discuss exact method to design a stochastic perturbation to stabilize a given unstable impulsive differential systems. The efficiency of the proposed results is illustrated by a numerical example.

1. Introduction

Recently, impulsive stochastic differential systems (ISDSs), which are subject to both impulsive effects and stochastic perturbations, have attracted considerable attention. As a result, many stability and stabilization results have been reported (see [1–11] and the reference therein). However, the previous works are mostly dedicated to moment stability. The only results on the almost sure exponential stability were established in [12–16] based on the assumption that the system is pth moment exponentially stable. Moreover, noise was often viewed as a perturbation with destabilization impact on the systems stability. In fact, there are several different concepts of stability in the literature on stochastic differential systems such as asymptotic stability in probability, pth moment exponential stability, and almost sure exponential stability, etc. For pth moment exponential stability, noise always plays a destabilizing role. That is, for an unstable system, there is no way to stabilize it by using noises in the sense of moment stability. It is well known that noise can be used to stabilize a given unstable deterministic differential systems in the sense of almost surely stability. The pioneering work was due to Hasminskii, who stabilized a two-dimensional linear system by using two white noise sources [17]. Following Hasminskii's work, there is an extensive literature concerned with the noise stabilization; see, for example, [18–30]. For deterministic impulsive differential systems (IDSs), one question arises: can noise make an exponentially unstable IDS exponentially stable?

Our aim in this paper is to seek a positive answer to this question. To explain this feature clearly, let us consider a simple linear scalar IDS

$$\begin{cases} \dot{y}(t) = ay(t), t \neq k, \\ \Delta y(k) = (c_k - 1)y(k^-), k = 1, 2, \dots, \end{cases} \tag{1}$$

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with initial value $y(0) = y_0 > 0$, where $a > 0$, $c_k \geqslant e^{-a/2}$, $\Delta y(k) = y(k) - y(k^-)$. The explicit solution to system (1) is

$$y(t) = y_0 e^{at} \prod_{0<k\leqslant t} c_k, \quad t \geqslant 0.$$

Hence

$$\lim_{t\to\infty} \frac{1}{t} \log(y(t)) = \lim_{t\to\infty} \frac{1}{t} \log(\prod_{0<k\leqslant t} c_k) + a \geqslant \frac{a}{2} > 0,$$

that is, the solution tends to infinity exponentially. To stabilize system (1), we perturb it by noise

$$\begin{cases} \mathrm{d}x(t) = ax(t)\mathrm{d}t + \sigma x(t)\mathrm{d}w(t), t \neq k, \\ \Delta x(k) = (c_k - 1)x(k^-), k = 1, 2, \ldots, \end{cases} \tag{2}$$

where $w(t)$ is a scalar Brownian motion and $\sigma > \sqrt{2a}$ represents the intensity of the noise. Given initial value $x(0) = x_0 > 0$, this system has explicit solution

$$x(t) = x_0 e^{(a-\frac{\sigma^2}{2})t+\sigma w(t)} \prod_{0<k\leqslant t} c_k, \quad t \geqslant 0,$$

which yields immediately that

$$\lim_{t\to\infty} \frac{1}{t} \log(x(t)) = \lim_{t\to\infty} \frac{1}{t} \log(\prod_{0<k\leqslant t} c_k) + a - \frac{\sigma^2}{2}$$

w.p.1 (with probability one). Let $e^{-\frac{a}{2}} \leqslant c_k < e^{\frac{\sigma^2}{4}-\frac{a}{2}}$. Then from the above inequality, one has

$$\lim_{t\to\infty} \frac{1}{t} \log(x(t)) \leqslant -(\frac{\sigma^2}{4} - \frac{a}{2}) \quad \text{w.p.1.}$$

This shows that for any $0 < \varepsilon < \frac{\sigma^2}{4} - \frac{a}{2}$, there exists a random variable $T_\varepsilon > 0$ such that

$$x(t) \leqslant e^{-(\frac{\sigma^2}{4}-\frac{a}{2}-\varepsilon)t}, \quad t \geqslant T_\varepsilon \quad \text{w.p.1,}$$

i.e., almost all sample paths of the solution will tend to the equilibrium point $x = 0$ exponentially fast. This implies that the perturbed system (2) becomes stable. In other words, the noise has almost surely stabilized the unstable system (1).

The main aim of this paper is to treat general nonlinear ISDSs. Suppose that we are given an unstable nonlinear IDS of the form

$$\begin{cases} \dot{x}(t) = f(x(t)), t \neq t_k, t \geqslant t_0 \\ \Delta x(t_k) = I_k(x(t_k)), k \in \mathbb{N}, \end{cases} \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ and $I_k : \mathbb{R}^n \to \mathbb{R}^n$ are Borel measurable functions. We aim to design a stochastic controller $g(x(t))\mathrm{d}w(t)$ such that the solution of ISDS (4) becomes almost surely exponentially stable. That is, we need to choose an appropriately function g such that the corresponding controlled system (4) will be almost surely exponentially stable.

This paper can be viewed as an extension of the works [17,21–24], which were concerned with almost sure exponential stability of stochastic differential systems (SDSs) without impulses. Compared with the previous works, the main contributions of this paper can be outlined as follows. First, impulsive effects are considered, which complicates the analysis. Second, the mutual restraints between the impulsive strength, the impulsive time interval, as well as the intensity of noise are derived. Third, the one-sided linear growth condition used in this paper is less conservative than that of the previous works.

The rest of the paper is organized as follows. Section 2 begins with some notation and system description. Section 3 develops some preliminary results that will play a basic role in this paper. The problems of almost sure exponential stability and stochastic stabilization by noise are discussed in Sections 4 and 5, respectively. An example is provided in Section 6, and conclusions are drawn in Section 7.

2. Preliminaries

Throughout this paper, unless otherwise specified, we shall use the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Denote by $\mathbb{E}[\cdot]$ the expectation operator with respect to the probability measure. Let $w(t) = (w_1(t), \ldots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{R}^{n\times m}$ the space of $n \times m$ real matrices, and I the identity matrix of appropriate dimension. For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm. If A is a vector or matrix, its transpose is denoted by A^T.