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Practical stability of nonlinear measure differential equations

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ABSTRACT

The paper investigates the practical stability of nonlinear measure differential equations. We first introduce the concepts of the practical stability and the practically asymptotic stability for measure differential equations. Then inspired by the proof method of the variational stability results in generalized ordinary differential equations, we provide the practical stability criteria for the measure system by using Lyapunov functions. Finally, as a special case, the practical stability results for discrete systems are given to illustrate our results.

1. Introduction

In this paper, we investigate the practical stability of measure differential equations. The usual hybrid systems separate continuous-time and discrete-time behaviors into two parts. Whereas measure differential equations integrate continuous-time and discrete-time dynamical behaviors into one system. Solutions to measure differential equations are given by functions of bounded variation and are not necessarily continuous. The discontinuous behavior is due to the presence of atoms in the measure. The study of measure differential equations first appeared in the context of optimal control problems where the constraint on the control input is that its integral remains bounded without any bounds on itself. Such problems are insoluble when the control input is taken as a Lebesgue-measurable function. Hence, using measures as control arises (see [1] and references therein). In addition, measure differential equations are also applied to frictional mechanics etc. [2].

From a mathematical point of view, measure differential equations are general and encompass ordinary differential equations, difference equations, impulsive differential equations (see [3,4]) and dynamic equations on time scales (see [5,6]) as special cases. And it is well known that these systems have been widely applied in modeling real-world problems. In addition, an interesting property of measure differential equations is that they allow very complex Zeno behaviors. In other words, the set of impulsive times is permitted to contain left and right accumulation points (see [7–9] and references therein). The so-called measure differential equations early investigated in [10,11] are actually still Zeno-free, since the discontinuity points in these systems are supposed to be isolated. This is the main distinction between the systems in [10,11] and the system we will consider in this paper. Beyond that, one can refer to the review paper [12], the monographs [1] and [13] for a detailed and complete introduction of measure differential equations.

Lyapunov stability pays attention to mathematically qualitative information of long-time behavior of the system. However, in many practical problems, one is interested in quantitative information concerning the system behavior such as estimation of trajectory bounds. For example, the desired state of a system may be unstable in the sense of Lyapunov, but the system may oscillate sufficiently near this state and the performance of the system is considered acceptable [14].

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To deal with these situations, the notion of the practical stability is proposed. Lakshmikantham first presents a systematic study of the theory of the practical stability [15]. Recently, much attention has been paid to the practical stability of several types of systems such as impulsive differential equations, fractional order dynamic systems, fuzzy differential equations and stochastic systems etc. [16–21]. However, to the best of our knowledge, no literatures have reported the practical stability of measure differential equations. This paper will try to fill in the gap.

The author studied the variational stability of generalized ordinary differential equations in the reference [22]. Such stability measures the distance of two solutions using the norm in the space of functions of bounded variation. However, just as Lyapunov stability, the variational stability also focus on mathematically qualitative information of the system. In this paper, we discuss the practical stability of measure differential equations, which concern the quantitative information of the system. The proof method here is inspired by the variational stability theory of generalized ordinary differential equations in [22]. Hence it is different from the proof line of the practical stability of the other types of differential equations in the existing references.

The paper is organized as follows. In Section 2, we give some concepts and results about Kurzweil–Stieltjes integral and measure differential equations. Main results are provided in Section 3. In Section 4, we relate discrete systems to measure differential equations and provide the practical stability criteria for discrete systems. Finally, some conclusions are drawn in Section 5.

2. Preliminaries

Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space with a norm $\|\cdot\|$, $\mathbb{R}_+$ the interval $[0, +\infty)$ and J an interval of $\mathbb{R}$. By regulated functions on J we mean that the left and right limits at each $t \in J$ exist whenever they can be defined. It is well known that the set of discontinuities for a regulated function is at most countable, but such a function need not be of bounded variation. By $G(J, \mathbb{R}^n)$ we denote the space of regulated functions from J to $\mathbb{R}^n$ with the topology of locally uniform convergence. The subspace of $G(J, \mathbb{R}^n)$ of left continuous functions is denoted by $G^-(J, \mathbb{R}^n)$.

Next we introduce Kurzweil–Stieltjes integral. This type of integral generalizes both Riemann and Lebesgue integrals (see e.g. [23]).

Consider a function $\delta : [a, +\infty) \to \mathbb{R}_+$, where $a \in \mathbb{R}$. A tagged partition of the interval $[\gamma, \vartheta] \subset [a, +\infty)$ with division points $\gamma = s_0 \le s_1 \le \cdots \le s_k = \vartheta$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i = 1, \ldots, k$, is called δ-fine if $[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$, $i = 1, \ldots, k$.

Definition 2.1 (*[22,24]*)**.** Consider the functions $f : [a, +\infty) \to \mathbb{R}^n$ and $g : [a, +\infty) \to \mathbb{R}$. We say that f is Kurzweil–Stieltjes integrable with respect to g over $[a, +\infty)$, if for every $\varepsilon > 0$, there are a function $\delta : [a, +\infty) \to \mathbb{R}_+$ and a positive number Δ such that

$$\left\| \sum_{j=1}^{m} f(\tau_j)(g(s_j) - g(s_{j-1})) - I_{[\gamma,\vartheta]} \right\| < \varepsilon$$

for every δ-fine tagged partition of a bounded interval $[\gamma, \vartheta] \supset [a, +\infty) \cap [-\Delta, \Delta]$ and some $I_{[\gamma,\vartheta]} \in \mathbb{R}^n$.

For a nondecreasing function g, we remark that the Kurzweil–Stieltjes integral exists whenever f is a regulated function. For basic properties of Stieltjes-type integrals one can refer to [25].

Consider the measure differential equation in the form

$$dx = f(t, x)dg, \quad t \ge t_0, \tag{1}$$

with the initial condition

$$x(t_0) = x_0, \tag{2}$$

where $t_0 \ge 0$; $g : [t_0, +\infty) \to \mathbb{R}$ is a left continuous nondecreasing function; $f : [t_0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is Kurzweil–Stieltjes integrable with respect to g; dx and dg denote the distributional derivatives of the state variable x and the function g in the sense of L. Schwartz, respectively.

As explained in [10], we have the following definition of solutions for the measure differential equation (1)–(2).

Definition 2.2. If there is a function $x : [t_0, +\infty) \to \mathbb{R}^n$ such that

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))dg(s), \quad t \ge t_0,$$

then x is called a solution of the system (1)–(2) on $[t_0, +\infty)$. Here the integral means Kurzweil–Stieltjes integral.

The following property of the indefinite Kurzweil–Stieltjes integral implies that solutions of measure differential equations are regulated functions (see Theorem 1.16 in [22] or Proposition 2.3.16 in [26]).