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Global stability of almost periodic solutions to monotone sweeping processes and their response to non-monotone perturbations

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We develop a theory which allows making qualitative conclusions about the dynamics of both monotone and non-monotone Moreau sweeping processes. Specifically, we first prove that any sweeping processes with almost periodic monotone right-hand-sides admits a globally exponentially stable almost periodic solution. And then we describe the extent to which such a globally stable solution persists under non-monotone perturbations.

1. Introduction

A perturbed Moreau sweeping process reads as

$$-\dot{x}(t) \in N_{C(t)}(x(t)) + f(t, x(t)), \tag{1}$$

where $N_C(x)$ is a so-called normal cone defined for closed convex $C \in \mathbb{R}^n$ as

$$N_C(x) = \begin{cases} \{\xi \in \mathbb{R}^n : \langle \xi, c - x\rangle \le 0, \ \text{for any } c \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C, \end{cases} \tag{2}$$

and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ (see [1–4]). The unboundedness of the right-hand-sides in (1) makes the classical theory of differential inclusions (see e.g. [5,6]) inapplicable. And despite numerous applications in elastoplasticity (see e.g. [7,8]) (as well as in problems of power converters [9] and crowd motion [10]), the theory of Moreau's sweeping processes is still in its

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infancy. Fundamental results on the existence, uniqueness and dependence of solutions on the initial data are proposed in Monteiro Marques [11, Ch. 3], Valadier [12], Castaing and Monteiro Marques [1], Adly–Le [13], Brogliato–Thibault [14], Krejci–Roche [15], Paoli [16]. Dependence of solutions on parameters is covered in Bernicot–Venel [17] and Kamenskii–Makarenkov [3]. The papers [1,3] also show the existence of T-periodic solutions for T-periodic in time (1). Optimal control problems for sweeping process (1) and equivalent differential equations with hysteresis operator are addressed in Edmond–Thibault [4], Adam–Outrata [18] (which also discusses applications to game theory), Brokate–Krejci [19]. Numerical schemes to compute the solutions of (1) are discussed through most of the papers mentioned above.

Much less is known about the asymptotic behavior as $t \to \infty$. The known results in this direction are due to Leine and van de Wouw [20,21], Brogliato [22], and Brogliato–Heemels [23]. Applied to a time-independent sweeping process (1) the statements of [20, Theorem 8.7] (or [21, Theorem 2]), [22, Lemma 2], and [23, Theorem 4.4] imply incremental stability (see Definition 2.1) and global exponential stability of an equilibrium, provided that

$$\langle f(t,x_1) - f(t,x_2), x_1 - x_2\rangle \geq \alpha\|x_1 - x_2\|^2, \quad \text{for some fixed } \alpha > 0 \text{ and for all } t \in \mathbb{R},\ x_1, x_2 \in \mathbb{R}^n. \tag{3}$$

In particular, the results of [20–23] do not impose any Lipschitz regularity on $x \mapsto f(t,x)$ and the derivative in (1) is a differential measure, which is capable to deal with solutions x of bounded variation.

This paper is motivated by sweeping processes (1) coming from models of parallel networks of elastoplastic springs (see e.g. Bastein et al. [7,8]), where the right-hand-sides are Lipschitz in all the variables. Here $C(t)$ represents the mechanical loading of the springs and $f(t,x)$ stands for those forces which influence the masses of nodes. Time-periodically changing C and f are most typical in laboratory experiments (see [7,24,25]). However, the different nature of $t \mapsto C(t)$ and $t \mapsto f(t,x)$ makes it most reasonable to not rely on the existence of a common period when the two functions receive periodic excitations, but rather to use a theory which is capable to deal with arbitrary different periods of $t \mapsto C(t)$ and $t \mapsto f(t,x)$. The goal of this paper is to develop such a theory.

Specifically, by assuming that both $t \mapsto C(t)$ and $t \mapsto f(t,x)$ are almost periodic, we establish global exponential stability of an almost periodic solution to a monotone sweeping process (14). The corresponding theory for differential equations is available e.g. in Trubnikov–Perov [26] and Zhao [27], that found numerous applications in biology. Moreover, we show that the almost periodic solution found preserves its stability under a wide class of non-monotone perturbations, which is known for differential inclusions with bounded right-hand-sides e.g. from Kloeden–Kozyakin [28] and Plotnikov [29].

The paper is organized as follows. Section 2 establishes (Theorem 2.1) the existence of solutions to (1) defined on the entire $\mathbb{R}$ under the assumption that both $t \mapsto C(t)$ and $(t,x) \mapsto f(t,x)$ are globally Lipschitz functions, but without any use of the monotonicity assumption (3). Note, that for any solution $x(t)$ of (1), one has $x(t) \in C(t)$, so any solution of (1) is uniformly bounded in the domain of its definition, if $C(t)$ is such. When the monotonicity assumption (3) holds, we have (Theorem 2.2) the uniqueness and global exponential stability of a solution defined on the entire $\mathbb{R}$. This result does not follow from [22,23], where the existence of an equilibrium is a consequence of the particular structure of the right-hand-sides. When both $C(t)$ and $f(t,x)$ are constant in t, the existence of an equilibrium to (1) formally follows from [20,21] which could transform into a solution on $\mathbb{R}$ when $C(t)$ and $f(t,x)$ are time-varying and globally bounded. We provide an independent proof because the proofs of [20, Theorem 8.7] and [21, Lemma 2] rely on Yakubovich [30, Lemma 2]. In turn, [30, Lemma 2] sends the reader to Budak [31, Theorem 2] for the most crucial step of the proof, which is compactness of a sequence $\{x_k\}_{k=1}^{\infty}$ of $C^0(\mathbb{R}, \mathbb{R}^n)$ solutions to (1) corresponding to a converging sequence of initial conditions. Even if one ignores verifying the regularity assumption of Budak [31, Theorem 2], this theorem provides a convergent subsequence on a finite interval and Yakubovich [30, Lemma 2] does not explain how the convergence gets extended to the entire $\mathbb{R}$.

Under the assumption that both $t \mapsto C(t)$ and $t \mapsto f(t,x)$ are almost periodic functions and $x \mapsto f(t,x)$ is monotone in the sense of (3), Section 3 shows (Theorem 3.1) that the unique global solution found in Section 2 is almost periodic. Here we follow the standard definitions (see e.g. Levitan–Zhikov [32, p. 1] or Vesely [33]) to introduce the concept of almost periodicity for set-valued functions and for the respective Bochner's theorem. The results of [32] and [33] are developed for functions with values in an arbitrary complete metric space and we take advantage of the completeness of the space of convex closed nonempty sets equipped with the Hausdorff metric (see e.g. Price [34]) to apply the concept of almost-periodicity to sweeping processes. The overall strategy of Section 3 originates from the corresponding theory available for differential equations (see e.g. Trubnikov–Perov [26]).

Section 4 considers a sweeping process (1) with a parameter ε under the assumption that the monotonicity condition (3) holds for $\varepsilon = \varepsilon_0$. When $\varepsilon = \varepsilon_0$, the sweeping process has a unique solution x_0 defined on $\mathbb{R}$ by Theorem 2.2. The result of Section 4 (Theorems 4.1 and 4.3) proves that the solutions to the perturbed sweeping process with $\varepsilon \neq \varepsilon_0$ and with an initial condition $x_\varepsilon(0) \in C(0)$ approach any given inflation of the solution x_0 (as it is termed in Kloeden–Kozyakin [28]) when the values of time become large and when ε approaches ε_0. Section 4.3 specifies the findings of Section 4 for the case where both $t \mapsto C(t)$ and $t \mapsto f(t,x,\varepsilon)$ are almost periodic in time, so that x_0 is almost periodic as well. Instructive examples of Section 4.4 illustrate the domains of applications of Theorems 4.1 and 4.3. Finally, Section 4.5 gives a brief outlook about the potential role of Theorems 4.1 and 4.3 in the analysis of the dynamics of networks of elastoplastic springs that motivated our study.

We note that condition (3) ensures that the sweeping process (1) is incrementally stable (see [20, Theorem 8.7], [21, Lemma 2], or Theorem 2.2 below), which concept currently attracts an increasing attention in the switched systems literature, see e.g. Lu–di Bernardo [35], Zamani–van de Wouw–Majumdar [36] and references therein. The source for