Nonlinear Analysis: Hybrid Systems 29 (2018) 373–382

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

Average break-even concentration in a simple chemostat model with telegraph noise☆

Chaoqun Xu [a], Sanling Yuan [a,*], Tonghua Zhang [b]

[a] College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
[b] Department of Mathematics, Swinburne University of Technology, Hawthorn, VIC. 3122, Australia

ARTICLE INFO

Article history:
Received 17 January 2017
Accepted 23 March 2018

Keywords:
Chemostat model
Telegraph noise
Markov chain
Average break-even concentration

ABSTRACT

In this paper, we aim to investigate the effect of telegraph noise on the continuous culture of microorganism in a chemostat. With the help of a finite-state Markov chain, we first construct a regime-switching chemostat model and then establish conditions for extinction and persistence of the microorganism. It is shown that the particular outcome of the chemostat is completely determined by a defined average break-even concentration λ^A: The microorganism becomes extinct in the chemostat when $\lambda^A > S^0$, the input substrate concentration; it will persist when $\lambda^A < S^0$. In the case of persistence, we also investigate some further dynamic behaviors of the solution including the recurrent level and eventually existent domain. The theoretical results are illustrated by simulations at the end of the paper.

1. Introduction

Let $S(t)$ be the concentration of substrate and $x(t)$ the concentration of microorganism in a chemostat at time t. Then the system of ordinary differential equations describing the continuous culture of microorganism in the chemostat takes the following form [1]:

$$\begin{cases} \dfrac{dS(t)}{dt} = (S^0 - S(t))D - \dfrac{mS(t)x(t)}{a + S(t)}, \\ \dfrac{dx(t)}{dt} = \left(\dfrac{mS(t)}{a + S(t)} - D\right) x(t), \end{cases} \tag{1.1}$$

where S^0 is the input substrate concentration, D is the washout rate. Monod functional response function $mS/(a + S)$ represents the substrate uptake rate of the microorganism, m is the maximal growth rate of microorganism, and a is called half-saturation constant, and furthermore all the parameters in (1.1) are positive. The global dynamics of the above system has been studied in [1,2]. In the case that $m \le D$, there only exists a washout equilibrium $E^0 = (S^0, 0)$ and it is globally asymptotically stable. In the case that $m > D$, the unique equilibrium E^0 is globally asymptotically stable when the break-even concentration $\lambda \equiv aD/(m - D) \ge S^0$; a positive equilibrium $E^* = (\lambda, S^0 - \lambda)$ appears and it inherits the global stability from E^0 when $\lambda < S^0$. Biologically speaking, if the maximal growth rate is not greater than the washout rate, the

☆ Research is supported by the National Natural Science Foundation of China (11671260, 11271260), Shanghai Leading Academic Discipline Project (XTKX2012), and Hujiang Foundation of China (B14005).

* Corresponding author.
E-mail address: sanling@usst.edu.cn (S. Yuan).

https://doi.org/10.1016/j.nahs.2018.03.007
1751-570X/

microorganism invariably becomes extinct in the chemostat. If the maximal growth rate is greater than the washout rate, the destiny of the microorganism is completely determined by its break-even concentration. For more studies on chemostat models, one can refer to Refs. [3–7] and the references cited therein.

Due to the continuous fluctuation in environment, the continuous culture of microorganism in the chemostat is neither autonomous nor deterministic. Hence, Imhof and Walcher [8] pointed that the consideration of stochastic effects is helpful in a proper understanding of a biological system, and a thorough study of stochastic chemostat models is justified. Recently, different versions of stochastic chemostat models have been proposed [9–14]. For example, in view of the fact that the maximal growth rate of microorganism is influenced by white noise, we replaced deterministic parameter m in model (1.1) by a random variable $m + \alpha\dot{B}(t)$ [9], resulting model

$$\begin{cases} \mathrm{d}S(t) = \left[(S^0 - S(t))D - \dfrac{mS(t)x(t)}{a+S(t)}\right]\mathrm{d}t - \dfrac{\alpha S(t)x(t)}{a+S(t)}\mathrm{d}B, \\ \mathrm{d}x(t) = \left(\dfrac{mS(t)}{a+S(t)} - D\right)x(t)\mathrm{d}t + \dfrac{\alpha S(t)x(t)}{a+S(t)}\mathrm{d}B, \end{cases} \tag{1.2}$$

where $B(t)$ is a standard Brownian motion, nonnegative constant α represents the intensity of the white noise. We found that if $m \le D$, the microorganism becomes extinct in chemostat; if $m > D$, for model (1.2) with small noise, there exists a stochastic break-even concentration

$$\tilde{\lambda} = \lambda + \frac{(S^0)^2\alpha^2}{2(a+S^0)(m-D)}$$

which can be seen as a critical value determining the persistence or extinction of the microorganism in the chemostat. This result was then improved by Zhao and Yuan [10].

Except for the white noise, telegraph noise is another type of environmental noise [15,16]. It consists of sudden instantaneous switching between two or more sets of parameter values in the underlying system corresponding to two or more different environments or regimes. This switching usually cannot be described by the traditional deterministic or Itô type stochastic models. But, some researchers pointed out that the sudden instantaneous switching existed in biological system could be formulated by a continuous time finite-state Markov chain [16–20]. For example, Takeuchi et al. [16] studied a Lotka–Volterra model with telegraph noise, in which the switching is governed by a two-state Markov chain. They showed the significant effect of telegraph noise on the system dynamics: Both subsystems develop periodically, but the switching system becomes neither permanent nor dissipative. In Ref. [17], Gray et al. considered the effect of telegraph noise on the prevalence of disease and proposed a regime-switching SIS epidemic model. They proved that the disease would go extinct almost surely if $T_0^S < 1$, while remain persist almost surely if $T_0^S > 1$.

Assume that the continuous culture of microorganism in the chemostat may switch between two different regimes (mark as 1 and 2) due to the variability of the environment, where regimes 1 and 2 represent the "good" and "bed" environments, respectively. There is a rather possible situation that the microorganism will persist in the chemostat with regime 1 (i.e., $\lambda_1 < S^0$) and extinct in the chemostat with regime 2 (i.e., $\lambda_2 > S^0$). Then two natural questions arise:

- Will the microorganism in the regime-switching chemostat persist or extinct?
- What does the destiny of microorganism in the regime-switching chemostat quantificationally depend on the probability distribution of the switching process?

Motivated by the above equations and published works, we will consider a regime-switching chemostat model corresponding deterministic model (1.1) to investigate the effect of the telegraph noise on dynamics of the chemostat. More precisely, in Section 2, we propose the regime-switching chemostat model and prove the global existence of the positive solution of the model. An average break-even concentration and the threshold dynamics are shown in Section 3. In Section 4, we show the asymptotic properties of the solution when the microorganism persists in the chemostat. Finally, numerical simulations and discussions are presented in the last section to close the paper.

2. Regime-switching chemostat model

Throughout this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$, which is increasing and right continuous and $\mathcal{F}_t$ contains all $\mathbb{P}$-null sets.

In the real-world environment, some environmental factors (such as temperature and PH value) may undergo an abrupt shift between different regimes due to the telegraph noise. It is worth noting that the maximal growth rate of microorganism is usually changed by the environmental factors. For instance, the maximal growth rate of microorganism at the high temperature will be much different from that at the low temperature. In this paper, we consider a case in which the maximal growth rate m may experience sudden instantaneous switching due to the telegraph noise. We then model the switching by a right-continuous Markov chain $r(t)$ taking values in finite-state space $\mathbb{S} = \{1, 2, \ldots, n\}$, resulting the following regime-switching chemostat model:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = (S^0 - S(t))D - \dfrac{m_{r(t)}S(t)x(t)}{a+S(t)}, \\ \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = \left(\dfrac{m_{r(t)}S(t)}{a+S(t)} - D\right)x(t). \end{cases} \tag{2.1}$$