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Deciding the boundedness and dead-beat stability of constrained switching systems

Matthew Philippe [a,*], Gilles Millerioux [b], Raphaël M. Jungers [a]

[a] ICTEAM Institute, Université Catholique de Louvain, Belgium
[b] Research Center for Automatic Control of Nancy (CRAN), Université de Lorraine, France

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ABSTRACT

We study computational questions related with the stability of discrete-time linear switching systems with switching sequences constrained by an automaton.

We first present a decidable sufficient condition for their boundedness when the maximal exponential growth rate equals one. The condition generalizes the notion of the irreducibility of a matrix set, which is a well known sufficient condition for boundedness in the arbitrary switching (i.e. unconstrained) case.

Second, we provide a polynomial time algorithm for deciding the *dead-beat* stability of a system, i.e. that all trajectories vanish to the origin in finite time. The algorithm generalizes one proposed by Gurvits for arbitrary switching systems, and is illustrated with a real-world case study.

1. Introduction

Switching systems are dynamical systems for which the state dynamics themselves vary between different operating modes according to a *switching sequence*. The systems under study in this paper are *discrete-time linear switching systems*. Given a set of N matrices $\mathbf{M} = \{A_1, \dots, A_N\} \subset \mathbb{R}^{n\times n}$, the dynamics of a discrete-time linear switching system are given by

$$x_{t+1} = A_{\sigma(t)} x_t,$$

where $x_0 \in \mathbb{R}^n$ is a given initial condition. The mode of the system at time t is $\sigma(t) \in \{1, \dots, N\}$. The *switching sequence* driving the system is the sequence of modes $\sigma(0), \sigma(1), \dots$, in time. Such systems are found in many practical and theoretical domains. For example they appear in the study of networked control systems [1,2], in congestion control for computer networks [3], in viral mitigation [4], as abstractions of more complex hybrid systems [5], and other fields (see e.g. [6–8] and references therein).

A large research effort has been devoted to the study of the stability and stabilization of switching systems (see e.g. [9,10,7,8,11,12]). The question of deciding the stability of a switching system is challenging and is known to be hard in general (see [6], Section 2.2, for hardness results).

In this paper, we first develop a sufficient condition for the boundedness of switching systems, that is, the existence of a uniform bound $K \geq 1$ such that for all switching sequences and all time $t \geq 0$,

$$\|A_{\sigma(t)} \cdots A_{\sigma(0)}\| \leq K. \tag{1}$$

* Corresponding author.
E-mail addresses: matthew.philippe@uclouvain.be (M. Philippe), gilles.millerioux@univ-lorraine.fr (G. Millerioux), raphael.jungers@uclouvain.be (R.M. Jungers).

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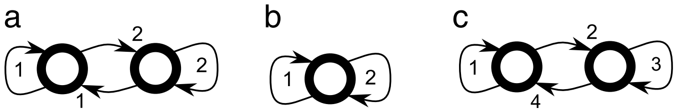

Fig. 1. Both automata on Fig. 1(a) and (b) accept *arbitrary switching sequences* on two modes. The third automaton (Fig. 1(c)) is on 4 modes. It does not generate arbitrary switching sequences, mode 1 must be followed by mode 1 or 2, mode 2 by mode 3 or 4, etc.

Second, we provide an algorithm for deciding when a system is *dead-beat* stable. This corresponds to the fact that there exists a time $T \geq 1$ such that, for all switching sequences, and all $t \geq T$, $A_{\sigma(t)} \cdots A_{\sigma(0)} = 0$. Both problems have been studied for *arbitrary switching systems* [13,6,14,15], for which the mode $\sigma(t)$ can take any value in $\{1, \ldots, N\}$ at any time. To the best of our knowledge, these studies have yet to be extended to switching systems with more general switching sequences, such as the ones studied in [16,10,4,17,11,9].

In this work, we allow for the definition of constraints on the switching sequences. These rules on the switching sequences are expressed through an *automaton*. Automata are common tools for the representation of admissible sequences of symbols (see [18], Section 1.3 for an introduction). An automaton is here represented as a *strongly connected* graph $\mathbf{G}(V, E)$, with a set of nodes V and edges E. The edges of this graph are both *directed* and *labeled*. An edge takes the form $(v, w, \sigma) \in E$, where v and w are respectively the *source* and *destination* nodes of the edge, and $\sigma \in \{1, \ldots, N\}$ is the *label*, taking its values in the set of modes of the switching system.

The edges E represent the possible time transitions of a switching system, and the nodes V need not be inherently associated with modes of the system. A switching sequence $\sigma(1), \sigma(2), \ldots$, of the system is then said to be *accepted* by $\mathbf{G}$ if there exists a path in $\mathbf{G}$ such that the sequence of labels *along the edges* of the path equals the switching sequence itself. Examples of such automatons, with their corresponding switching rules, are presented in Fig. 1. Note that, in general, there can be several automata that represent a same set of switching rules. Because it is not relevant to our purpose, we do not specify an initial or final node for the paths—in this we differ from the classical definition of an automaton ([18], Section 1.3).

Given a graph $\mathbf{G}(V, E)$ on N labels and a set of N matrices $\mathbf{M}$, we define the *constrained switching system* $S(\mathbf{G}, \mathbf{M})$ as the following discrete-time linear switching system:

$$x_{t+1} = A_{\sigma(t)} x_t, \quad \sigma(0), \ldots, \sigma(t) \text{ is accepted by } \mathbf{G}. \tag{2}$$

Arbitrary switching systems are special cases of constrained switching systems. Their switching rule can be represented by the automata alike those of Fig. 1(a) and (b), or by any *path-complete graph* (see [16]).

The boundedness and stability properties of a system $S(\mathbf{G}, \mathbf{M})$ are tightly linked to its *constrained joint spectral radius*. This concept was introduced by X. Dai [19] in 2012 for the stability analysis of constrained switching systems. The CJSR of the system $S(\mathbf{G}, \mathbf{M})$ is defined as follows:

$$\hat{\rho}(S) = \lim_{t\to\infty} \max\left\{ \left\| A_{\sigma(t-1)} \cdots A_{\sigma(0)} \right\|^{1/t} : \sigma(0), \ldots, \sigma(t-1) \text{ is accepted by } \mathbf{G} \right\}. \tag{3}$$

When $\mathbf{G}$ allows for *arbitrary switching*, the CJSR is equal to the *joint spectral radius* (JSR) of the set $\mathbf{M}$, which was introduced by Rota and Strang in 1960 (see [6] for a monograph on the topic).

The constrained joint spectral radius is the maximal exponential growth rate of a system. Its value reflects the stability properties of a system $S(\mathbf{G}, \mathbf{M})$. If $\hat{\rho}(S) < 1$, the system is both asymptotically and exponentially stable (see [19]-Corollary 2.8). If $\hat{\rho}(S) > 1$, the system possesses an unbounded trajectory whose growth rate is exponential. The last case $\hat{\rho}(S) = 1$ is more complicated. The system is not asymptotically stable, but may be bounded or not depending on its parameters.

If S is an arbitrary switching system with $\hat{\rho}(S) = 1$, then there exists a condition guaranteeing its boundedness. This condition is the *irreducibility* of $\mathbf{M}$.

Definition 1. A set $\mathbf{M} \subset \mathbb{R}^{n\times n}$ of matrices is *irreducible* if for any non-trivial linear subspace $\mathcal{X} \subset \mathbb{R}^n$, i.e. with $0 < \dim(\mathcal{X}) < n$, there is a matrix $A \in \mathbf{M}$ such that $A\mathcal{X} \not\subset \mathcal{X}$. That is, the matrices in $\mathbf{M}$ do not share a common non-trivial invariant subspace of $\mathbb{R}^n$.

Proposition 1.1 *(e.g., [6], Theorem 2.1)***.** *If a set of matrices* $\mathbf{M}$ *with joint spectral radius equal to* 1 *is irreducible, then the arbitrary switching system on the set* $\mathbf{M}$ *is bounded.*

The irreducibility property is known to be decidable (see [20] and references therein). Moreover, irreducibility implies the existence of *Barabanov* norms for arbitrary switching systems. The existence of such norms is very useful for the stability analysis of arbitrary switching systems [21,6,22].

The first part of this paper is focused around providing a proper generalization of both irreducibility and Proposition 1.1 for constrained switching systems.[1] As shown in Example 1, Proposition 1.1 does not generalize directly to constrained switching systems.

[1] Preliminary results on this question were presented in the conference paper [23].

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