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Optimal switching for hybrid semilinear evolutions

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ABSTRACT

We consider the optimization of a dynamical system by switching at discrete time points between abstract evolution equations composed by nonlinearly perturbed strongly continuous semigroups, nonlinear state reset maps at mode transition times and Lagrange-type cost functions including switching costs. In particular, for a fixed sequence of modes, we derive necessary optimality conditions using an adjoint equation based representation for the gradient of the costs with respect to the switching times. For optimization with respect to the mode sequence, we discuss a mode-insertion gradient. The theory unifies and generalizes similar approaches for evolutions governed by ordinary and delay differential equations. More importantly, it also applies to systems governed by semilinear partial differential equations including switching the principle part. Examples from each of these system classes are discussed.

1. Introduction

We consider hybrid dynamical systems on some infinite (or finite) dimensional space Z and a finite set of modes $\mathcal{M}$. For a given family $\{A^j\}_{j\in\mathcal{M}}$ of densely defined linear operators on Z, families of nonlinear functions $\{f^j\}_{j\in\mathcal{M}}$ and $\{g^{j,j'}\}_{j,j'\in\mathcal{M}\times\mathcal{M}}$ on Z and a finite time horizon $[0, T]$ with initial condition $z(0) = z_0 \in Z$ the dynamics are governed by abstract continuous time evolution equations combined with discrete events involving state resets

$$\dot{z} = A^j z + f^j(z), \quad z = g^{j,j'}(z^-)$$

whenever the mode $j \in \mathcal{M}$ is held constant or whenever j with associated state z^- is switched to the new mode $j' \in \mathcal{M}$ with new state z at switching times $(\tau_k)_{k\in\mathbb{N}} \subseteq [0, T]$, respectively. Supposing that the sequence of switching times $(\tau_k)_k$ and the modal sequence $(j_k)_k$ are subject to our control and that we have a cost function $J = J(z)$ integrating running and switching cost associated to the respective continuous or discrete evolution, we may consider the minimization of J over any such sequences of finite length as an optimal control problem. The precise setting and main hypotheses are introduced in Section 2.

This and variants of this optimal control problem have been extensively addressed for *ordinary differential equations* (ODEs), e.g., based on dynamic programming principles [1,2], non-smooth programming [3], control parametrization enhancing techniques [4] and relaxation techniques [5,6]. Moreover, if the modal sequence $(j_k)_k$ is a-priori fixed, the control problem reduces to switching-time optimization and can be solved using gradient-based methods [7–10]. The latter approach has also been extended using gradients with respect to mode-insertions into a given sequence [8]. Switching-time optimization and mode-insertions can be combined to conceptual algorithms to tackle the original problem [11,12]. We refer to [13] for a more detailed survey of available results for the ODE case.

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Much less work has been done for similar optimal control problems in context of ordinary *delay differential equations* (DDEs) and *partial differential equations* (PDEs). Such problems arise for example in optimal control of gas networks, where switching of valves is an essential part of the control mechanism for the gas flow governed by algebraically coupled PDEs on a graph representing the network of pipes [14,15]. Switching-time optimization has been considered for ordinary DDEs in [16,17] and, when switching only affects boundary data, for scalar hyperbolic PDEs in the semilinear case [18] and in the non-linear case [19]. In a more abstract fashion based on semigroup theory covering both, certain DDEs and PDEs, dynamic programming extends to problems when A^j is a generator of a strongly continuous semigroup independent of j and switching only affects the non-linear perturbation [20]. In the same setting, relaxation techniques can sometimes be applied [21].

Our contribution in this paper is to extend the concept of switching-time optimization and mode-insertion from ODE problems in [8] to the abstract setting of non-linearly perturbed strongly continuous semigroups. Unlike in [8], we consider non-autonomous dynamics, state-resets at switching times and include switching costs. Moreover, among switching of the non-linear perturbation, our theory explicitly considers switching of the generators, which (in non-trivial cases) cannot be handled with the results available in the literature so far. This allows – under certain technical restrictions – the treatment of switching, e.g., the delay parameter of a DDEs or switching the principle part of a PDE in the hybrid dynamical system represented by the above equations. Our analysis focuses on the differentiability properties of the cost function and the representation of the derivative using solutions to appropriate adjoint problems. The analysis of gradient-descent algorithms using such derivative information as well as applications for example to gas network optimization will be considered in future work.

In Section 2 we introduce our abstract problem setting including the hypotheses concerning the regularity of the system parameters. In Section 3 we consider differentiation of the costs with respect to the switching times for a given mode sequence. In Section 4 we discuss differentiation of the costs with respect to the insertion of a new mode into a given sequence of modes. In Section 5 we show that one can recover the result of [8] for the ODE case from our theory under rather mild technical assumptions on the system parameters. Moreover, we show that the results can be used for example to obtain efficient gradient-representations of integro-type DDE and that the theory is consistent with stability analysis for a PDE switching between a transport equation and a diffusion equation.

2. Notation, basic hypotheses and preliminaries

In the presentation of our results, we mainly use standard notion from the theory of strongly continuous semigroups as for example in [22]. Nevertheless, for clarity, we mention the following notation and conventions used in context of a Banach space Z. We denote by Z^* the topological dual space of Z and for every $z^* \in Z^*$ the dual pairing by $\langle z^*, z\rangle_{Z^*,Z} := z^*(z)$ for all $z \in Z$. A map $f : Z \to Z$ is called *differentiable in* $z \in Z$, if it is Fréchet-differentiable in z, that is, if there is a linear bounded operator $Df(z)(.)$ on Z, such that

$$\lim_{\|h\|_Z \searrow 0} \frac{\|f(z+h) - f(z) - Df(z)(h)\|_Z}{\|h\|_Z} = 0.$$

Finally, f is called *differentiable*, if it is differentiable in every $z \in Z$ and *continuously differentiable*, if $Df(.)$ is continuous as an operator from Z into the space of bounded linear operators on Z. If $D \subseteq \mathbb{R}^n$ is open and $f : D \to \mathbb{R}^m$ is continuously differentiable, then we say f is continuously differentiable on the closure $\overline{D}$, if both f and f' can be continued as continuous functions to $\overline{D}$ and again $f'(x)$ is called the derivative of x for all $x \in \overline{D}$.

Our basic hypotheses in this paper are as follows.

(A1) Z is a reflexive Banach space and $z_0 \in Z$, $\mathcal{M}$ is a finite set and $j_0 \in \mathcal{M}$.
(A2) A^j is for every $j \in \mathcal{M}$ the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S^j(t)\}_{t\geq 0}$ on Z with domain $D(A^j) \subseteq Z$.
(A3) For every $i, j \in \mathcal{M}$ let $f^i \in C^1([0,\infty) \times Z, Z)$ and let $g^{i,j} : Z \to Z$ be a given map.
(A4) $z_0 \in D(A^{j_0})$ and the map $g^{i,j}$ is continuously differentiable for all $i, j \in \mathcal{M}$ with $i \neq j$, satisfying the inclusion $g^{i,j}(D(A^i)) \subseteq D(A^j)$.

The hybrid semilinear evolutions are specified as follows: Given a fixed $N \in \mathbb{N}_0$, a sequence of modes $j = (j_n)_{n=0,\dots,N} \subseteq \mathcal{M}$ and a monotonically increasing, but not necessarily strictly increasing sequence of switching times $\tau = (\tau_n)_{n=0,\dots,N+1} \subseteq [0,\infty)$, we consider dynamics of the form

$$\begin{aligned} \dot z(t) &= A^{j_n} z(t) + f^{j_n}(t, z(t)), && n \in \{0,\dots,N\},\ t \in (\tau_n, \tau_{n+1}),\\ z(\tau_n) &= g^{j_{n-1},j_n}(z^-(\tau_n)), && n \in \{1,\dots,N\},\\ z(\tau_0) &= z_0. \end{aligned} \tag{1}$$

A map $z : [\tau_0, \tau_{N+1}] \to Z$ is called a *mild solution* to (1), if, for all $n \in \{0,\dots,N\}$, there are functions $z^n : [\tau_n, \tau_{n+1}] \to Z$ satisfying the following conditions:

(i) z^n is the only element of $C([\tau_n, \tau_{n+1}], Z)$ satisfying the variation of constants formula

$$z^n(t) = S^{j_n}(t-\tau_n) z_0^n + \int_{\tau_n}^{t} S^{j_n}(t-s) f^{j_n}(s, z^n(s))\, ds \quad \forall\, t \in [\tau_n, \tau_{n+1}],$$