Nonlinear Analysis: Hybrid Systems 20 (2016) 72–81

Contents lists available at ScienceDirect

Nonlinear Analysis: Hybrid Systems

journal homepage: www.elsevier.com/locate/nahs

On existence of nonlinear measure driven equations involving non-absolutely convergent integrals

Yueju Cao [a,b], Jitao Sun [a,*]

[a] Department of Mathematics, Tongji University, Shanghai, 200092, China

[b] College of Science, Northwest A&F University, Yangling, Shaanxi, 712100, China

ARTICLE INFO

Article history:
Received 30 May 2015
Accepted 12 November 2015

Keywords:
Measure differential equations
Kurzweil integral
Regulated functions
Measure of noncompactness

ABSTRACT

In this paper, we study the existence of solutions for nonlinear measure driven equations in Kurzweil integral setting. Firstly, some new results on Hausdorff measure of noncompactness in the space of regulated functions are established. Then some existence criteria of the measure system are provided by applying Hausdorff measure of noncompactness and a corresponding fixed point theorem. The results in this paper improve and generalize those well known in the literature. Finally, an example is given to illustrate our results.

1. Introduction

In this paper, we consider the following nonlinear measure driven integral system:

$$x(t) = x_0 + (\text{HLS}) \int_0^t f(s, x(s))dg(s), \quad t \in J, \tag{1}$$

where $J = [0, a]$ with $a > 0$; the state variable $x(\cdot)$ takes values in a Banach space X; $g : J \to \mathbb{R}$ is a nondecreasing function continuous from the left; $f : J \times X \to X$; (HLS) $\int_0^{\cdot}$ denotes the Henstock–Lebesgue–Stieltjes integral, a kind of non-absolutely convergent integral as a special case of Kurzweil integral, which will be specified later. The system (1) can be related to the following measure driven differential problem

$$\begin{aligned} &dx(t) = f(t, x(t))dg(t), \quad t \in J, \\ &x(0) = x_0, \end{aligned} \tag{2}$$

where dx and dg denote the distributional derivatives of the solution and the function g, respectively [1,2]. However, it have to be stated that the equivalence between the sets of solutions of the two problems is a delicate matter, which relies on the chosen definition of solutions to (2) (see [1,2] and references therein).

Measure driven differential equations are also called measure differential equations, which arise in many areas of applied mathematics such as nonsmooth mechanics, game theory etc. (see [3–7] and references therein). dg in (2) can be identified with a Lebesgue–Stieltjes measure. Based on different g, measure differential equations cover some well-known cases. When g is an absolutely continuous function, a step function, or the sum of an absolutely continuous function with a step function, the system corresponds to ordinary differential equations, difference equations or impulsive differential

* Corresponding author. Tel.: +86 21 65983241x1307; fax: +86 21 65981985.
E-mail address: sunjt@sh163.net (J. Sun).

http://dx.doi.org/10.1016/j.nahs.2015.11.003
1751-570X/

equations respectively. Further, compared to usual impulsive systems (see [8–10]), measure differential equations, as another approach to develop impulsive action in dynamic systems, admit the discontinuous paths that may exhibit infinitely many discontinuities in a finite interval. This kind of important property makes measure differential equations can possibly model some non-classical problems like the quantum effects and Zeno trajectories (see [11,12]).

Measure differential equations were investigated early by [13–16]. One can refer to the review paper [17] for a complete introduction of measure differential systems. Recently, Tvrdý [18] has introduced the so-called regulated functions and used the Kurzweil–Stieltjes integral to study solutions in the class of regulated functions for general first-order linear systems of equations with measures. The papers [19–22] investigated nonlinear measure functional differential equations by applying the method of generalized ordinary differential equations. However, all these papers focused on measure equations in $\mathbb{R}^n$ space. To the best of our knowledge, few literatures have been devoted to measure differential equations in infinite dimensional spaces except [2,23]. By using Hausdorff measure of noncompactness, the paper [2] discussed the existence of solutions for nonlinear measure driven system (1). Although some properties of Hausdorff measure of noncompactness in the space of regulated functions were provided in [2], those properties are not intrinsic for the space of regulated functions since the function g in the system was involved. Moreover, the proof was not in detail and questionable (see Theorem 5 in [2]). Under Lipschitz-type conditions, [19,20] studied the retarded version of nonlinear measure driven system by transforming measure equations into generalized ordinary differential equations when $X = \mathbb{R}^n$. In addition, the authors in [23] investigated the existence of mild solutions for abstract semilinear measure driven system in Lebesgue integral setting. In this paper, we first establish some useful properties of Hausdorff measure of noncompactness in the space of regulated functions, which are different from those in [2]. Based on these properties and a corresponding fixed point theorem, we get the distinct existence criteria from those in [2] for measure driven system (1). In addition, without any assumptions of Lipschitz-type as those in [19,20], a similar analysis to the system (1) can lead to the existence result of nonlinear measure retarded equations.

This paper is organized as follows. In Section 2, we recall some concepts and basic results about Henstock–Lebesgue–Stieltjes integral together with regulated functions. And we establish some important properties of Hausdorff measure of noncompactness on regulated functions, which will be used throughout this paper. Main results are provided in Section 3. An example that illustrates our results is presented in Section 4. Finally, some conclusions are drawn in Section 5.

2. Preliminaries

In this section, we recall some concepts and basic results about Henstock–Lebesgue–Stieltjes integral as well as regulated functions. In addition, we establish some important properties of Hausdorff measure of noncompactness in the space of regulated functions.

Let X be a Banach space with the norm $\|\cdot\|$ and $J = [0, a]$ a closed interval of the real line. A function $f : J \to X$ is called regulated on J, if the limits

$$\lim_{s\to t^-} f(s) = f(t^-), \quad t \in (0, a] \quad \text{and} \quad \lim_{s\to t^+} f(s) = f(t^+), \quad t \in [0, a)$$

exist and are finite (see [24,25]). The space of regulated functions $f : J \to X$ is denoted by $G(J; X)$. It is well known that the set of discontinuities of a regulated function is at most countable and that the space $G(J; X)$ is a Banach space endowed with the norm $\|f\|_\infty = \sup_{t\in J} \|f(t)\|$ (see [24]).

The finite sets $d = \{t_0, t_1, \ldots, t_m\}$ of points in the closed interval J such that $0 = t_0 < t_1 < \cdots < t_m = a$ are called partitions of J. Let $\delta > 0$, we say that a partition of J is δ-fine, if for every $i = 1, 2, \ldots, m$, we have $|t_i - t_{i-1}| < \delta$. Moreover, we call a pair $(\tau_i, [t_{i-1}, t_i])$ to be a tagged interval, where $\tau_i \in [t_{i-1}, t_i]$ is a tag of $[t_{i-1}, t_i]$. Consider a function $\delta : J \to \mathbb{R}^+$ (called a gauge on J). A tagged partition of the interval J with division points $0 = t_0 < t_1 < \cdots < t_m = a$ and tags $\tau_i \in [t_{i-1}, t_i]$, $i = 1, \ldots, m$, is called δ-fine if $[t_{i-1}, t_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i))$, $i = 1, \ldots, m$. For more details, the readers can consult [26].

Definition 2.1 (*See [2]*)**.** Let $g : J \to \mathbb{R}$. A function $f : J \to X$ is said to be Henstock–Lebesgue–Stieltjes integrable with respect to (w.r.t. for short) g on J (shortly, HL-Stieltjes integrable) if there exists a function denoted by (HLS) $\int_0^{\cdot} : J \to X$ such that, for every $\varepsilon > 0$, there is a gauge δ_ε on J with

$$\sum_{i=1}^{m} \left\| f(\xi_i)(g(t_i) - g(t_{i-1})) - \left((\text{HLS}) \int_0^{t_i} f(s)dg(s) - (\text{HLS}) \int_0^{t_{i-1}} f(s)dg(s) \right) \right\| < \varepsilon$$

for every δ_ε-fine partition $\{([t_{i-1}, t_i], \xi_i) : i = 1, 2, \ldots, m\}$ of J.

The HL-Stieltjes integrability is preserved on all sub-intervals of J. The function $t \mapsto$ (HLS) $\int_0^t f(s)dg(s)$ is called the HL-Stieltjes primitive of f w.r.t. g on J (we refer to [18] or [26] for finite dimensional space X).

Proposition 2.2 (*See [2]*)**.** *Let $g : J \to \mathbb{R}$ and $f : J \to X$ be HL-Stieltjes integrable w.r.t. g. If g is regulated, then so is the primitive $h : J \to X$, $h(t) =$ (HLS) $\int_0^t f(s)dg(s)$ and*

$$h(t^+) = h(t) + f(t)[g(t^+) - g(t)], \quad t \in [0, a),$$
$$h(t^-) = h(t) - f(t)[g(t) - g(t^-)], \quad t \in (0, a].$$