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Letter

Time–space dependent fractional boundary layer flow of Maxwell fluid over an unsteady stretching surface

Shengting Chen, Liancun Zheng *, Bingyu Shen, Xuehui Chen

School of Mathematics and Physics, University of Science and Technology Beijing, Beijing 100083, China

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ABSTRACT

Fractional boundary layer flow of Maxwell fluid on an unsteady stretching surface was investigated. Time–space dependent fractional derivatives are introduced into the constitutive equations of the fluid. We developed and solved the governing equations using explicit finite difference method and the L1-algorithm as well as shifted Grünwald–Letnikov formula. The effects of fractional parameters, relaxation parameter, Reynolds number, and unsteadiness parameter on the velocity behavior and characteristics of boundary layer thickness and skin friction were analyzed. Results obtained indicate that the behavior of boundary layer of viscoelastic fluid strongly depends on time–space fractional parameters. Increases of time fractional derivative parameter and relaxation parameter both cause a decrease of velocity while boundary layer thickness increase, but the space fractional derivative parameter and fractional Reynolds number have the opposite effects.

Much attention has been paid to the study of boundary layer flow induced by continuously stretching sheets submerged in a quiescent or moving fluid due to its important applications in industries (e.g., copper wire's drawing, annealing, and thinning, aerodynamic extrusion of plastic sheets and fibers, paper production, crystal growing, and glass blowing). In magnetic field and thermal radiation field, the dissipative boundary layer flow on a nonlinearly stretching sheet was studied by Kumbhakar et al. [1]. With convective boundary condition, the three dimensional radiative flow of Maxwell fluid over an inclined stretching surface was investigated by Ashraf et al. [2]. In a constantly applied magnetic field, the steady mixed convection stagnation point flow of an incompressible Oldroyd-B fluid over the stretching sheet was analyzed by Sajid et al. [3]. Likewise, the problems of unsteady boundary layer were studied widely. Analyses of the unsteady magnetohydrodynamic (MHD) boundary layer flow and heat transfer of an incompressible rotating viscous fluid over a continuously stretching sheet were performed by Abbas et al. [4]. A numerical analysis of the structure of an unsteady boundary layer flow and heat transfer of a dusty fluid over an exponentially stretching sheet subjected to suction was done by Pavithra et al. [5]. The effects of a chemical reaction on an unsteady flow of a micropolar fluid over a stretching sheet embedded in a non-Darcian porous medium were studied by Srinivas et al. [6].

The viscoelastic materials have the properties of both viscosity and elasticity. Scott Blair [7] proposed a fractional viscoelastic fluid constitutive model using the relation

$$\tau(t) = \nu \frac{\mathrm{d}^{\alpha}\sigma(t)}{\mathrm{d}t^{\alpha}}, \tag{1}$$

where ν is a constant, $\tau(t)$ is the stress, $\sigma(t)$ stands for the strain rate, and α is a constant ranging from 0 to 1.

Traditional researches on viscoelastic fluid were carried on in the cases with the governing equations being linear. Caputo and Mainardi [8,9] have shown that results obtained in their analysis were in good agreement with experimental results when fractional derivative is used to describe the viscoelastic materials. El-Shahed et al. [10] obtained exact analytic solutions of a few cases in Navier–Stokes equations with time fractional derivative. By applying the He's homotopy perturbation method (HPM) and variational iteration method (VIM), Khan et al. [11] studied the Navier–Stokes equations with fractional orders. Since viscoelastic fluid shows properties of both elasticity and viscosity, many fractional models have been proposed to characterize the constitutive relationship between viscous stress and the strain rate for viscoelastic materials. MHD flow of an incompressible generalized Oldroyd-B fluid caused by an accelerating plate was studied by Zheng et al. [12], and they obtained the exact solutions for velocity and shear stress in terms of Fox H-function. A number of the recent works can be also found in Refs. [13–19].

* Corresponding author.
E-mail address: liancunzheng@ustb.edu.cn (L. Zheng).

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However, the authors of Refs. [13–19] have ignored the nonlinear term of convection and have dealt with special simple cases where the governing equations are linear. Solutions were obtained with the help of Laplace transform, Fourier Sine transform and finite Hankel transform. To our knowledge, no report has been made for fractional viscoelastic fluid boundary layer flow with non-linear term of convection considered.

In this paper, the governing equations of fractional viscoelastic fluid induced by an unsteady stretching surface are developed and solved coupled with the unsteady boundary using the explicit finite difference and L1-algorithm as well as shifted Grünwald–Letnikov formula (approximations for fractional derivatives). The effects of involved parameters on velocity field, boundary layer thickness, and skin friction are then analyzed and discussed.

Considered an unsteady boundary layer flow of the Maxwell fluid over an unsteady stretched sheet, which can be depicted by the time–space dependent fractional derivatives, the shear stress can be expressed in the following form

$$\tau = \bar{\mu}\frac{\partial^\beta u}{\partial y^\beta}. \tag{2}$$

By ignoring the pressure gradient, the governing equations take the following forms

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3}$$

$$\left(1+\lambda D_t^\alpha\right)\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \bar{v}\frac{\partial}{\partial y}\left(D_y^\beta u\right), \tag{4}$$

where D_t^α and D_y^β are fractional calculus operator based on Caputo's definition and Riemann–Liouville's definition respectively [20]

$$D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha-1)}\int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}\mathrm{d}\tau, \quad 0<\alpha<1, \tag{5}$$

$$D_y^\beta f(y) = \frac{1}{\Gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \frac{f(\xi)}{(y-\xi)^\beta}\mathrm{d}\xi, \quad 0<\beta<1, \tag{6}$$

where $\Gamma(\cdot)$ is the Gamma function, u and v stand for the horizontal velocity and vertical velocity respectively, $\bar{v} = \bar{\mu}/\rho$ is the fractional kinematics viscosity of the fluid (in $\mathrm{m}^{1+\beta}/\mathrm{s}$), $\bar{\mu}$ is the fractional viscosity coefficient (in $\mathrm{kg/m}^{2-\beta}/\mathrm{s}$), ρ is the constant density of the fluid (in $\mathrm{kg/m}^3$), and λ is the fractional relaxation time (in $1/\mathrm{s}^\alpha$).

It is assumed that the fluids are static on the plate at first, suddenly the sheet achieves a horizontal velocity U_w along the x-axis. The shear stress results in the movement of the fluids. The governing equations are given by Eqs. (3) and (4) and satisfy the boundary conditions

$$\begin{aligned}
&u(0,y,t)=0, \quad u(x,0,t)=U_\mathrm{w},\\
&u(x,y,t)\to 0 \quad y\to L, \quad u(x,y,t)\to 0 \quad x\to\infty,\\
&v(x,0,t)=v(0,y,t)=0, \quad v(x,y,t)\to 0 \quad x\to\infty,\\
&v(x,y,t)\to 0 \quad y\to L,
\end{aligned} \tag{7}$$

where the unsteady stretching velocity U_w is horizontal and depend on time and space. It is assumed to be

$$U_\mathrm{w} = ax/(1-bt). \tag{8}$$

Applying the following non-dimensional quantities

$$\begin{aligned}
&x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad u^* = \frac{u}{aL}, \quad v^* = \frac{v}{aL},\\
&t^* = at, \quad \lambda^* = a^\alpha\lambda
\end{aligned} \tag{9}$$

and ignoring the dimensionless mark "*" for brevity, we can derive the dimensionless motion equations as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{10}$$

$$\left(1+\lambda D_t^\alpha\right)\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re_\beta}\frac{\partial}{\partial y}\left(D_y^\beta u\right). \tag{11}$$

The initial and boundary conditions are

$$\begin{aligned}
&u(0,y,t)=0, \quad u(x,0,t)=x/(1-St),\\
&u(x,y,t)\to 0 \quad y\to 1, \quad u(x,y,t)\to 0 \quad x\to\infty,\\
&v(x,0,t)=v(0,y,t)=0, \quad v(x,y,t)\to 0 \quad x\to\infty,\\
&v(x,y,t)\to 0 \quad y\to 1,
\end{aligned} \tag{12}$$

where $S = b/a$ is the unsteadiness parameter, and $Re_\beta = \frac{aL^{\beta+1}}{\bar{v}}$ is the general fractional Reynolds number.

We first discretize space and time into grid points and time instants, letting $x_i = ih_x\ (i=0,1,2,\ldots)$, $y_j = jh_y\ (j=0,1,2,\ldots)$, and $t_n = k\tau\ (k=0,1,2,\ldots)$, where h_x, h_y and τ are the spatial and temporal steps respectively.

Adopting the L1-algorithm [21] into the unsteady term, we can obtain

$$\begin{aligned}
D_t^\alpha u(x,y,t)\Big|_{x_i,y_j}^{t_k} &= \frac{1}{\tau^\alpha\Gamma(2-\alpha)}\\
&\times\left[c_0u_{i,j}^k - \sum_{l=1}^{k-1}(c_{j-k-1}-c_{j-k})u_{i,j}^l - c_{j-1}u_{i,j}^0\right]\\
&+O(\tau^{2-\alpha}), \quad 0<\alpha<1,
\end{aligned} \tag{13}$$

where the diffusion term is approximated using the shifted Grünwald–Letnikov formula [22]

$$\begin{aligned}
D_y^\gamma u(x,y,t)\Big|_{x_i,y_j}^{t_k} &= \frac{1}{h_y^\gamma}\sum_{p=0}^{j+1} w_p u_{i,j-p+1}^k + O(h_y),\\
&1<\gamma<2.
\end{aligned} \tag{14}$$

Here the coefficients are defined as

$$c_k = (k+1)^{1-\alpha} - k^{1-\alpha}, \tag{15}$$

$$w_0 = 1, \quad w_k = \left(1-\frac{\beta+1}{k}\right)w_{k-1}, \quad k=1,2,\ldots. \tag{16}$$

Introducing the Euler backward difference scheme into the first-order time derivative, we have

$$\frac{\partial u(x_i,y_j,t_k)}{\partial t} = \frac{u(x_i,y_j,t_k)-u(x_i,y_j,t_{k-1})}{\tau} + O(\tau). \tag{17}$$

The explicit finite difference approximations for Eqs. (10) and (11) are

$$\left(\frac{1}{h_y}\right)v_{i,j}^{k+1} - \left(\frac{1}{h_y}\right)v_{i,j-1}^{k+1} = \left(\frac{1}{h_x}\right)u_{i-1,j}^k - \left(\frac{1}{h_x}\right)u_{i,j}^k, \tag{18}$$

$$\begin{aligned}
&\frac{u_{i,j}^{k+1}-u_{i,j}^k}{\tau} + \frac{\lambda}{\tau^\alpha\Gamma(2-\alpha)}\\
&\quad\times\left[c_0\left(u_{i,j}^k-u_{i,j}^{k-1}\right) - \sum_{l=1}^{k-1}(c_{k-l-1}-c_{k-l})\left(u_{i,j}^l-u_{i,j}^{l-1}\right)\right]\\
&\quad+u_{i,j}^k\frac{u_{i,j}^k-u_{i-1,j}^k}{h_x} + v_{i,j}^k\frac{u_{i,j}^k-u_{i,j-1}^k}{h_y}\\
&= \frac{1}{Re_\beta h_y^{\beta+1}}\sum_{p=0}^{j+1} w_p\cdot u_{i,j-p+1}^k.
\end{aligned} \tag{19}$$