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Quaternion frames and fractal surface as tools to control orientation of a spacecraft

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ABSTRACT

Reorientation of an object's (spacecraft) problem is formulated in details in the $SO(3, R)$ and $SU(2)$ groups matrix terms using the most optional math tool of exceptional algebra of quaternion numbers. A thorough analysis of the two approaches is made resulting in original formulas linking parameters of the assigned object's consequent 3D rotations with a single rotation about a unit vector pointing the instant rotation axis, respective operational technology described with relevant examples. It is also demonstrated that an axial quaternion frame admits fractalization so that the reorientation problem is reduced to deformations of the sub-geometric fractal surface.

1. Introduction

There are few math-tech methods to ensure orientation of an object moving or immobile in space. Two evident branches of the assigned orientation problem may be distinguished, they are: (i) a series of subsequent several-angles rotation, (ii) a one-angle rotation about an instant axis. Mixed variants may exist, but implying more cumbersome calculations, hence apparently less productive, they are not considered here. Usually the orientation tasks are relevant with computations over three-dimensional (3D) flat space modeling a local domain of the physical space; but since the math methods noticeably differ, various matrix algebra elements are engaged.

If magnitudes involved in calculations are regarded as real number objects, then the both techniques (i) and (ii) should be preferably based on the vector rotation group $SO(3, R)$. Then in the case (i) the solution is reduced to a set of simple (plane) rotations by Euler (or Krylov, or others) angles about selected axes, in the case (ii) a non-trivial problem of searching for the instant axis of a single rotation is to be solved.

It is widely known though that quaternion (Q-) numbers perfectly fit for the spacecraft orientation tasks (see e.g. [1,2]) due to the fact that three Q-vector units represent math models of three mutually orthogonal gyroscope axes (Q-frame). This simplifies calculations, especially for the case (ii) since not only the vector $SO(3, R)$, but also the spinor $SU(2)$ reflection group can be used rotation group, though a clear interconnection between parameters of respective matrix representations and the relevant operational technique are hardly found in literature. This study aims to fill the gap with the help of the simplest math means. By the way we encounter an unexpected possibility to split any axial vector, and a Q-frame as a whole into 2D vector-covector constituents of fractal dimension ½ (provided any of 3D dimensions is taken for 1). The fractalization technique [3], mathematically non-trivial and much less known (so formerly ignored) endows all algebraic objects and actions with distinct geometric sense. Detailed description of this technique applied to the reorientation problem is another purpose of this study. In Section 2 parameters of an object's orientation in 3D space are described, and a review of the math procedure within technique (i) is given. In Section 3 the Q-algebra is shortly represented with analysis of form-invariance of its multiplication law. Section 4 is devoted to solution of the spacecraft reorientation problem within approach of a Q-frame single rotation. In Section 5 a 2D fractal space "underlying" the 3D space is introduced, and a simplified reorientation procedure is described having its "joy-stick" analog in the fractal space. In Section 6 a compact discussion of the method practical implementation and relevant perspectives concludes the study.

2. Orientation parameters of a space object and description of its 3D rotations

If a coordinate system adjusted to some mechanical situation is given, then orientation of a spacecraft (a rigid body) in the space is defined by three angles between the coordinate axes and unit vectors of a movable frame "frozen in" the object (usually matched with its symmetry). An Earth observer would use the globe-based spherical coordinates normally implying two (right-handed) Cartesian directions in the horizontal plane, those to the North along a meridian $\mathbf{e}_1$ and along a parallel $\mathbf{e}_2$; the orthogonal third one is zenith direction $\mathbf{e}_3$. The set $\mathbf{e}_k$ (small Latin indices run through 1, 2, 3) is taken for constants. Then the orientation of a spacecraft bearing a frame $\mathbf{e}_{k'}$, (with $\mathbf{e}_{1'}$ along

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it, $\mathbf{e}_{2'}$ a transverse one, $\mathbf{e}_{3'}$ along gravity) is determined by three angles: "yaw" ψ, the angle between $\mathbf{e}_1$ and $\mathbf{e}_{1'}$ (rotation about $\mathbf{e}_3$); "roll" ϕ, angle $\mathbf{e}_2$-$\mathbf{e}_{2'}$ (rot. about $\mathbf{e}_1$); "pitch" θ, angle $\mathbf{e}_3$- $\mathbf{e}_{3'}$ (rot. about $\mathbf{e}_2$). Within these notations the object's orientation in the space is described by the matrix equation

$$\mathbf{e}_{n'} = O_{n'k}\mathbf{e}_k \tag{1}$$

where $O_{n'k}$ (we'll also use simpler notation O) is a 3×3-matrix belonging to the special orthogonal group $O \in SO(3, R)$, so its properties are

$$O_{n'k}O_{n'm} = \delta_{km}, \quad \det O_{n'k} = 1, \tag{2}$$

repeated indices imply summation, i.e. $\mathbf{e}_{1'} = O_{1'k}\mathbf{e}_k = O_{1'1}\mathbf{e}_1 + O_{1'2}\mathbf{e}_2 + O_{1'3}\mathbf{e}_3$, δ_{km} is the Kronecker symbol (here it is 3D unit matrix).

Outlined in Section 1 technique (i) demands that the matrix $O_{n'k}$ be represented as a product of simple rotations [irreducible representations of $SO(3, R)$], each performed about a frame's unit vector; a special notation for such matrix is O_n^α, lower index is a number of the rotation axis (the frame's unit vector), upper index is the rotation angle. A simple rotation e.g. changing the "yaw" is given by the matrix

$$O_3^\psi \equiv \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

Direct reorientation problem, i.e. reaching object's assigned orientation, can be solved by a sequence of simple rotations mathematically described by a sequent multiplication of matrices (3). This problem has no unique solution since the group $SO(3, R)$ is not commutative, i.e. different multiplication order of the matrices (3) with the same parameters (angles) generally gives different result, e.g. products $O = O_3^\psi O_2^\phi O_1^\theta$ and $O' = O_1^\theta O_2^\phi O_3^\psi$ are normally different $O \neq O'$. Vice versa, different orders of the matrix product with other parameters may yield the same result, e.g. products $O = O_3^\psi O_2^\phi O_1^\theta$ and $O' = O_1^{\theta'} O_2^{\phi'} O_3^{\psi'}$ may be equal $O = O'$. Thus the reorientation problem in the technique (i) is ambiguous. Moreover, this introduces non-uniqueness in the solution of the inverse problem, search of angles providing actual object's orientation. Indeed, decomposition of an arbitrary $SO(3, R)$ matrix into irreducible representations can be made in different ways; for example one can check up that the following representation of the orthogonal 3×3-matrix evidently belonging to the group $SO(3, R)$ (each arbitrary element $x, \ y, \ z$ equals its adjoint) as a product of three simple rotation is not unique

So the question of the optimal choice of the set of simple rotation angles arises in solution of the inverse problem.

Moreover, transit to technique (ii) in terms of $SO(3, R)$-matrices is known in theory of matrices (see e.g. [4]) an uneasy algebraic task, comprising search of the operator's O eigenvector X with unit eigenvalue $O_{k'n}X_n = \delta_{kn}X_n$ (vector X directing the axis of single rotation), followed by vague procedure of finding value of the rotation angle. Fortunately all these difficulties are successfully and transparently coped with the help of quaternion algebra tools successfully used for solving navigation and orientation problems from the 60-s of XX century.

3. Quaternion SO(3,R) approach to the reorientation problem

Quaternion (Q-) number is a math object of the form[1]

$$q \equiv a + b_k\mathbf{q}_k, \tag{5}$$

$a, b_k \in \mathbf{R}$; the component $a = a\cdot 1$ is called the scalar part (the scalar unit 1 is normally omitted), $b_k\mathbf{q}_k$ is the vector part ($\mathbf{q}_k$ are three imaginary vector units). The properties of the Q-number are determined by the multiplication law for its units

$$1\ \mathbf{q}_k = \mathbf{q}_k 1 = \mathbf{q}_k, \quad \mathbf{q}_k\mathbf{q}_l = -\delta_{kl} + \varepsilon_{klm}\mathbf{q}_m, \tag{6}$$

where ε_{jkl} is completely antisymmetric 3D discriminant tensor (Levi-Chivita symbol).

Comparison, addition, and subtraction properties of Q-numbers are similar to those of complex numbers. But Eq. (6) state that the quaternion multiplication is not commutative (though it is still associative). A Q-number can be conjugated, it has the norm and modulus

$$\bar{q} \equiv a - b_k\mathbf{q}_k, \quad q\bar{q} = |q|^2, \quad |q| = \sqrt{q\bar{q}} = \sqrt{a^2 + b_k b_k}. \tag{7}$$

Eq. (7) lead to definition of the inverse Q-number, hence, to right and left quaternion division

$$q^{-1} = \bar{q}/|q|^2 \rightarrow (q_1/q_2)_{right} = q_1\bar{q}_2/|q_2|^2, \quad (q_1/q_2)_{left} = \bar{q}_2 q_1/|q_2|^2. \tag{8}$$

The norm of a product of two Q-numbers $q_1 = a + b_k\mathbf{q}_k$ and $q_2 = c + d_k\mathbf{q}_k$ equals to product of norms of the multiplies

$$|q|^2 = |q_1q_2|^2 = (q_1q_2)\overline{(q_1q_2)} = q_1q_2\bar{q}_2\bar{q}_1 = q_1\bar{q}_1q_2\bar{q}_2 = |q_1|^2|q_2|^2. \tag{9}$$

In the developed form Eq. (9) demonstrates the identity of four squares: a product of two sums of four squares is again a sum of four squares

$$\begin{aligned} &(ac - b_1d_1 - b_2d_2 - b_3d_3)^2 + (ad_1 + cb_1 + b_2d_3 - b_3d_2)^2 \\ &\quad + (ad_2 + cb_2 + b_3d_1 - b_1d_3)^2 + (ad_3 + cb_3 + b_1d_2 - b_2d_1)^2 \\ &= =(a^2 + b_1^2 + b_2^2 + b_3^2)(c^2 + d_1^2 + d_2^2 + d_3^2). \end{aligned} \tag{10}$$

The "squares identities" are distinctive feature of only four exclusive algebras of real, complex, quaternion, and octonion numbers (the last one based on 8 units is non-associative).

Q-vector units $\mathbf{q}_n$ have geometric sense important for applications: they behave as vectors forming a Cartesian frame in 3D space. Indeed, the simplest 2×2-matrix representation of the units[2]

$$\mathbf{q}_1 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{q}_2 = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{q}_3 = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{11}$$

explicitly demonstrates the "vector product" property appropriate to axial vectors

$$\mathbf{q}_3 = \mathbf{q}_1\mathbf{q}_2, \quad \mathbf{q}_2 = \mathbf{q}_3\mathbf{q}_1, \quad \mathbf{q}_1 = \mathbf{q}_2\mathbf{q}_3 \tag{12}$$

directing a Cartesian coordinate axes. It is one of the most important property of Q-algebra is that its multiplication law Eq. (6) keeps its form if the Q-units $\mathbf{q}_k$ are transformed

$$\mathbf{q}_{n'} = O_{n'k}\mathbf{q}_k \tag{13}$$

exactly as the polar basic vectors $\mathbf{e}_k$ in Eq. (1); this means that a Q-frame subject to a rotation of the type (13) is a new triad of orthonormal Q-vectors. Thus, all $SO(3, R)$ formulas and conclusions together with the problems mentioned in Section 2 are valid for frames described by quaternions. However, happily the Q-math provides a different (and simpler) solution of the orientation problem due to the "enigmatic" procedure of "two-side multiplication" of a quaternion. Indeed, one can demonstrate (e.g. [2]) that the combination URU^{-1} of quaternions R and $U \equiv a + b\mathbf{q}$ conically rotates vector part of R about the unit Q-vector $\mathbf{q}$ at the angle $2\arctan(b/a)$. Detailed analysis of this type of description of rotations is in the next section.

[1] Quaternion units originally denoted (by Hamilton) as $1, \ \mathbf{i}, \ \mathbf{j}, \ \mathbf{k}$ here are given in compact vector format $1, \ \mathbf{q}_n$.

[2] We use here 2×2-matrix representation of quaternions without loss of generality.