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Exponentiated modified Weibull extension distribution

Ammar M. Sarhan [*,1], Joseph Apaloo

Department of Mathematics, Statistics & Computer Science St. Francis Xavier University, Antigonish NS, Canada B2G 2W5

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ABSTRACT

A new modified Weibull extension distribution is proposed by Xie et al. [20]. Recently, El-Gohary et al. [9] proposed a new distribution referred to as the generalized Gompertz distribution. In this paper, we propose a new model of a life time distribution that mainly generalizes these two distributions. We refer to this new distribution as the exponentiated modified Weibull extension distribution. This distribution generalizes, in addition to the above two mentioned distributions, the exponentiated Weibull distribution, the generalized exponential and the generalized Rayleigh distributions. Parameter estimation of the four parameters of this distribution is studied. Two real data sets are analyzed using the new distribution, which show that the exponentiated modified Weibull extension distribution can be used quite effectively in fitting and analyzing real lifetime data.

1. Introduction

Weibull distribution is one of the most commonly used lifetime distributions in reliability and lifetime data analysis. It is flexible in modeling failure time data, as the corresponding hazard rate function can be increasing, constant or decreasing. But in many applications in reliability and survival analysis, the hazard rate function can be of bathtub shape. The hazard rate function plays a central role to the work of reliability engineers, see Lai and Xie [13] and Bebbington et al. [3,4] and references therein. Models with a bathtub hazard rate function are needed in reliability analysis and decision making when the life time of the system is to be modeled.

Many parametric probability distributions have been introduced to analyze sets of real data with bathtub-shaped hazard rates. The bathtub-shaped hazard function provides an appropriate conceptual model for some electronic and mechanical products as well as the lifetime of humans. Some work on parametric probability distributions with bathtub-shaped hazard rate functions have been considered by different authors. The exponential power distribution was suggested by Smith and Bain [19], and it was studied by Leemis [15]. A four parameter distribution was proposed by Gaver and Acar [10]. A similar distribution with increasing, decreasing, or bathtub-shaped hazard rate has been considered by Hjorth [12]. An exponentiated Weibull distribution with three parameters was suggested by Mudholkar and Srivastava [17].

Chen [8] discussed an interesting two-parameter model that can be used to model bathtub-shaped hazard rate function. Though this distribution has only two parameters, it shows a bathtub shaped hazard rate. However, it is not flexible because it does not include a scale parameter. Xie et al. [20] proposed a new modified extension of the Weibull distribution with a bathtub-shaped hazard rate function. We refer to this extension as the new modified Weibull extension (MWE) distribution. The MWE generalizes the two-parameter model discussed by Chen [8] and it includes one scale parameter and two shape parameters.

The cumulative distribution function of the MWE distribution [20] is

$$F_{\mathrm{MWE}}(x;\lambda,\alpha,\beta)=1-\exp\{\lambda\alpha[1-e^{(x/\alpha)^{\beta}}]\},\quad x\geq 0,\ \lambda,\alpha,\beta>0. \tag{1}$$

Setting $\alpha=1$ in (1), we get the cdf of the two-parameter distribution discussed by Chen [8] as a sub-model of the MWE distribution.

Since 1995, exponentiated distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. A good review of some of these models is presented by Pham and Lai [18].

Mudholkar and Srivastava [17] proposed the exponentiated Weibull distribution (EW or EW (σ,β,γ)), with the following cdf:

$$F_{\mathrm{EW}}(x;\sigma,\beta,\gamma)=\left[1-\exp\left\{-\left(\frac{x}{\sigma}\right)^{\beta}\right\}\right]^{\gamma},\quad x\geq 0,\ \sigma,\ \beta,\ \gamma>0. \tag{2}$$

El-Gohary et al. [9] proposed the exponentiated Gopertz distribution, and referred to it as the generalized Gompertz (GG) distribution, whose cdf is of the form

$$F_{\mathrm{GG}}(x;\lambda,c,\gamma)=\left[1-\exp\left\{\frac{\lambda}{c}(1-e^{cx})\right\}\right]^{\gamma},\quad \lambda,\ c,\ \gamma>0. \tag{3}$$

* Corresponding author. Tel.: +1 902 266 6423; fax: +1 902 4945130.
E-mail addresses: asarhan0@yahoo.com, asarhan@stfx.ca (A.M. Sarhan), japaloo@stfx.ca (J. Apaloo).
[1] Permanent address: Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.

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Though some distributions such as the exponentiated Weibull distribution, modified Weibull extension and generalized Gompertz distribution are known to have bathtub-shaped hazard rate, they may not be able to give a good bathtub shape of the hazard rate (see [21]). However, there are fewer models whose hazard rate curves are similar to the actual bathtub shape. Zhang et al. [21] discussed the parametric analysis of some models which exhibit a good bathtub shaped hazard rate.

In this paper, we propose a new distribution which generalizes the above three distributions, with the hope that it will attract many applications in different fields such as reliability, lifetime data analysis, and others. This new distribution exhibits a good bathtub shaped hazard rate which is very similar to the actual bathtub curve. Mainly, we consider our new distribution as a generalization of the MWE distribution. On the other hand, it can be considered as a generalization of the GG distribution as well as the EW distribution. We will refer to this new distribution as the exponentiated modified Weibull extension (EMWE) distribution. The EMWE distribution contains as special sub-models, in addition to the above three mentioned distributions, many distributions such as exponential, generalized exponential, Weibull, Rayleigh, generalized Rayleigh, among others. Due to the flexibility of the EMWE in accommodating different forms of the hazard rate functions, especially the ones which have wide flat portions, it seems to be a suitable distribution that can be used in a variety of problems for fitting reliability data. The EMWE distribution is not only convenient for fitting bathtub-shaped hazard rate data but it is also suitable for testing goodness-of-fit of some special sub-models such as the MWE, EW, and GG distributions.

The rest of the paper is organized as follows. In Section 2, we introduce the EMWE distribution, discuss some special sub-models and provide its cumulative distribution function (cdf), the probability density function (pdf) and the hazard function. A formula for generating random samples from the EMWE distribution is also given in Section 2. Section 3 discusses some important statistical properties of the EMWE distribution such as the ordinary moments and measures of skewness and kurtosis. Section 4 discusses the parameter estimation process using maximum likelihood estimates. Two applications to real data are provided in Section 5. Section 6 concludes the paper. The paper also contains an Appendix giving technical details.

2. The EMWE distribution

The cdf of the exponentiated modified Weibull extension distribution with four parameters $\theta=(\alpha,\beta,\lambda,\gamma)$, abbreviated as EMWE distribution, is

$$F(x;\theta)=[1-e^{\lambda\alpha(1-e^{(x/\alpha)^\beta})}]^\gamma, \quad \lambda,\ \alpha,\ \beta,\ \gamma>0,\ x\ge 0. \tag{4}$$

The probability density function of the EMWE(θ) distribution is

$$f(x;\theta)=\lambda\beta\gamma\left(\frac{x}{\alpha}\right)^{\beta-1} e^{(x/\alpha)^\beta+\lambda\alpha(1-e^{(x/\alpha)^\beta})}[1-e^{\lambda\alpha(1-e^{(x/\alpha)^\beta})}]^{\gamma-1},$$
$$\lambda,\alpha,\ \beta,\ \gamma>0,\ x\ge 0. \tag{5}$$

The hazard rate function of the EMWE(θ) distribution is

$$h(x;\theta)=\frac{\lambda\beta\gamma\left(\frac{x}{\alpha}\right)^{\beta-1} e^{(x/\alpha)^\beta+\lambda\alpha(1-e^{(x/\alpha)^\beta})}}{[1-e^{\lambda\alpha(1-e^{(x/\alpha)^\beta})}]^{1-\gamma}+e^{\lambda\alpha(1-e^{(x/\alpha)^\beta})}-1}, \quad \lambda,\ \alpha,\ \beta,\ \gamma>0,\ x\ge 0. \tag{6}$$

One can see from Fig. 1 that the hazard function: (1) takes a bathtub shape if either $\gamma<1$ whatever the value of β or $\beta<1$ whatever the value of γ and (2) is increasing if $\beta\ge 1$ and $\gamma\ge 1$. The bathtub-shaped curve of Fig. 1(b) has quite a long flat part which is very similar to the actual bathtub shaped curve. The EMWE distribution has two scale (α, λ) and two shape (β,γ) parameters and generalizes several well known distributions. The following is a list of well known sub-models of the EMWE distribution.

When the scale parameter α becomes very large or tends to infinity while $\alpha^{\beta-1}/\lambda$ remains constant, the EMWE distribution reduces to the exponentiated Weibull with scale parameter $\alpha^{\beta-1}/\lambda$ and shape parameters β and $\sigma=\alpha^{\beta-1}/\lambda$, say EW($\sigma$,$\beta$,$\gamma$).

When $\beta=1$ and α becomes very large or tends to infinity while $\alpha^{\beta-1}/\lambda$ remains constant, the EMWE distribution reduces to the generalized exponential distribution with a scale parameter $1/\lambda$ and shape parameters γ, say GE ($1/\lambda$,γ), see Gupta and Kundu [11].

When $\beta=1$, the EMWE distribution reduces to the generalized Gompertz distribution with scale parameters λ and $c=1/\alpha$ and shape parameter γ, say GG (λ,c,γ), see El-Gohary et al. [9].

When $\gamma=1$ and $\alpha=1$, the EMWE reduces to the distribution in Chen [8] with shape parameters λ and β.

The pdf of the EMWE distribution (5) can be written as a linear combination of the pdf of MWE distribution. For $\gamma>0$, a series expansion for $(1-w)^{\gamma-1}$, for $|w|<1$, is

$$(1-w)^{\gamma-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\gamma)}{\Gamma(\gamma-j)j!}w^j, \tag{7}$$

where $\Gamma(.)$ is the gamma function. Since for $x>0$, $S_{MWE}(x)=S_{MWE}(x;\lambda,\alpha,\beta)=\exp\{\lambda\alpha[1-e^{(x/\alpha)^\beta}]\}<1$, then using the series expansion (7) in (5), we obtain

$$f(x;\theta)=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\gamma+1)}{\Gamma(\gamma-j)(j+1)!}f_{\text{MWE}}(x;(j+1)\lambda,\alpha,\beta). \tag{8}$$

When γ is a positive integer, the index j in (8) stops at $\gamma-1$. The linear combination (8) enables us to obtain some mathematical properties of EMWE directly from those of the MWE distribution such as, the moments, the moment generating function, characteristic function.

There are many softwares such as MATLAB, MAPLE and MATHEMATICA that can be used to compute (8) numerically.

Advantage 1: One of the advantages of the EMWE distribution is that it has a closed form cdf, which can be used to generate random numbers from it by using the following simple formula:

$$X=\alpha\left\{-\ln\left[1-\frac{1}{\alpha\lambda}\ln(1-U^{1/\gamma})\right]\right\}^{a/\beta}, \tag{9}$$

where U is a uniformly distributed random variable on (0, 1) interval. The formula (9) can be used to generate random samples from a wide set of sub-models of the EMWE distribution such as the exponential, generalized exponential, Rayleigh, generalized Rayleigh, Weibull, modified Weibull extension, Exponentiated Weibull, Gompertz and generalized Gompertz distributions.

Interpretation: When γ is a positive integer, the EMWE(θ) distribution can be interpreted as the lifetime distribution of a parallel system consisting of γ independent and identical units whose lifetime follows the MWE (λ,α,β) distribution.

3. The moments, skewness and kurtosis

The k-th ordinary moment of the EMWE distribution can be written as linear combination of those for the MWE distribution. Let $\mu_k(\theta)$ and $\mu_{k,\text{MWE}}(\lambda,\alpha,\beta)$ be the k-th moments of the EMWE and MWE distributions, respectively, then

$$\mu_k(\theta)=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(\gamma+1)}{\Gamma(\gamma-j)(j+1)!}\mu_{k,\text{MWE}}((j+1)\lambda,\alpha,\beta). \tag{10}$$