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The probability distribution of maintenance cost of a system affected by the gamma process of degradation: Finite time solution

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ABSTRACT

The stochastic gamma process has been widely used to model uncertain degradation in engineering systems and structures. The optimization of the condition-based maintenance (CBM) policy is typically based on the minimization of the asymptotic cost rate. In the financial planning of a maintenance program, however, a more accurate prediction interval for the cost is needed for prudent decision making. The prediction interval cannot be estimated unless the probability distribution of cost is known. In this context, the asymptotic cost rate has a limited utility.

This paper presents the derivation of the probability distribution of maintenance cost, when the system degradation is modelled as a stochastic gamma process. A renewal equation is formulated to derive the characteristic function, then the discrete Fourier transform of the characteristic function leads to the complete probability distribution of cost in a finite time setting. The proposed approach is useful for a precise estimation of prediction limits and optimization of the maintenance cost.

1. Introduction

1.1. Background

The gamma process is an example of an analytically tractable stochastic cumulative process that is widely used to model degradation processes, such as corrosion, creep and wear, in engineering systems, structures and components [1]. The reliability of degrading systems is managed by engineers through Condition-based maintenance (CBM), which is one of many alternative policies for the maintenance. The basic idea of a CBM policy is to inspect and quantify the degradation at regular intervals, so that the system can be repaired or replaced preventively before it fails in a catastrophic manner.

The theory of gamma processes provides an analytical framework for predicting the reliability and estimating the maintenance cost, including costs of inspection, repair or replacements, and consequences of the failure. This probabilistic model can be subsequently used for cost optimization by appropriately choosing the inspection interval and preventive maintenance (PM) criterion.

Several variations of CBM models have been discussed in the literature, depending on whether or not the inspection schedule is periodic, inspection tools are perfect, failure is detected immediately, or repair duration is negligible. Abdel-Hameed [2] and Park [3] presented models of periodic CBM of components subjected to gamma process degradation. Other applications include recession of coastal cliffs [4], deterioration of coating on steel structures [5], concrete structure degradation [6], breakwater and sea floor protection [7,8], and wall thinning of pipes in nuclear power plants [9]. The model of non-periodic CBM was presented by Grall et al. [10] and that of imperfect inspection by Kallen and van Noortwijk [11]. Castanier et al. [12] studied such a maintenance policy in which both the future maintenance (replacement or imperfect repair) and the inspection schedule depend on the magnitude of degradation. The case of delayed repair is analyzed in [13]. Optimization of inspection and repair for the Wiener and gamma processes of degradations was discussed in [14]. A comprehensive review of the gamma process model and its applications can be found in a recent review article of van Noortwijk [1].

In most of the literature, CBM optimization is based on minimization of the asymptotic cost rate, because the renewal theorem provides a simple expression for its computation [15]. The asymptotic cost rate is equal to the expected cost in one renewal cycle divided by its expected length or duration. The asymptotic formulation has a universal appeal in the maintenance optimization literature, because it basically reduces the stochastic renewal process model to the first failure problem. However, this simplification may not be realistic for many engineering systems with a relatively short and finite operating life.

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Table 1
Notation & acronym.

c_I, c_F, c_P	Unit inspection cost, CM cost, and PM cost
$q_{CM}(k)$, $q_{PM}(k)$	Probability of a renewal cycle ending at time k by CM and that by PM
$q_T(k)$	Probability mass function of one renewal cycle length
w_F, w_P	Degradation threshold of failure and that of PM
u_∞	Asymptotic cost rate
β, α	Scale and shape parameter of the gamma degradation process
δ	Inspection interval
μ_C	Expected cost in one renewal cycle
μ_T	Expected length of one renewal cycle
$C(\tau)$	Cost in the time interval $(0,\tau]$
CM	Corrective maintenance
PM	Preventive maintenance
CBM	Condition based maintenance
E[*]	Expected value
P{*}	Probability
T	Length of one renewal cycle
$U(\tau)$, $V(\tau)$, $\sigma(\tau)$	Expected value, second moment about the origin, and standard deviation of $C(\tau)$
$\{W(\tau), \tau = 0, 1, \ldots\}$	Discrete stochastic process of degradation
ρ	Unit cost
n_c	An integer such that $n_c\rho$ is the upper bound of $C(\tau)$
FT	Fourier transform
PDF	Probability density function
PMF	Probability mass function

The financial risk assessment of a CBM program deals with different issues, such as how much capital is required to implement a CBM policy and what is the residual risk after the implementation of a CBM policy. Financial risk measures, such as Value-at-Risk (VaR), need to be evaluated for prudent decision-making. To address these questions, it is clear that complete probability distribution of maintenance cost is required, which would allow to compute accurate prediction limits of cost and assess the financial risk.

1.2. Research objectives and approach

The evaluation of expected cost is reasonable for finding an optimal maintenance policy among a set of alternatives in a relative sense. This approach is not informative enough to enable the estimation of financial risk measures, such as percentiles of the cost, also known as Value-at-Risk (VaR).

The key objective of this paper is to present the derivation of probability distribution of maintenance cost of a system subjected to stochastic degradation, e.g., gamma process.

The proposed solution is based on the fact that the characteristic function of a continuous/discrete random variable is the inverse Fourier transform of its probability density/mass function. Therefore, a renewal equation is firstly formulated to evaluate the characteristic function. Then, the Fourier transform of the characteristic function is computed, which leads to complete probability distribution of cost in a finite time setting. Once the cost distribution is derived, financial risk measures, such as VaR, are easily calculated.

1.3. Organization

This paper is organized as follows. Section 3 presents the terminology, assumptions and basic concepts of cost analysis of a CBM policy. A brief sketch of the gamma process model of degradation is presented in Section 3. This model is utilized to derive the distribution of the renewal cycle length. Section 4 summarizes the derivation of statistical moments of maintenance cost in a finite time horizon. In Section 5, ideas presented in Section 4 are used to formulate a renewal equation for the characteristic function, and its Fourier transform leads to the probability distribution of maintenance cost. Illustrative numerical examples are presented in Section 6, and conclusions are summarized in the last section. The notations and the acronyms used in this paper are listed in Table 1.

2. Condition-based maintenance (CBM)

2.1. Terminology and assumptions

The operation of a system is considered in a finite time horizon $(0,\tau)$, where τ is also referred to as the operating life of the system. The time span is discretized in some suitable unit as $0, 1, 2, \ldots, k, \ldots, \tau$. This unit can be years, months, days or hours depending on the system under consideration. The total degradation, $W(k)$, at any arbitrary time k is a non-decreasing stochastic gamma process.

The component is inspected at times $t_1, t_2, \ldots, t_j (\le \tau)$, where j denotes the number of inspection. Under a periodic inspection policy, the inspections are carried out at a fixed interval δ, such that $t_j = j\delta$. During an inspection, it is assumed that the degradation is perfectly detectable and its magnitude is quantified without any sizing error.

A failure occurs at a time k when degradation exceeds a critical threshold w_F. The failure event is self-announced, i.e., it is immediately detectable. The maintenance following a failure is referred to as a corrective maintenance (CM), as shown by Fig. 1(a). At a time of inspection (t_j), if the degradation exceeds a threshold, w_P ($< w_F$), but does not exceed w_F, the preventive maintenance (PM) will commence (Fig. 1(b)). If $W(t_j) \le w_P$, no action is taken. The time required for PM and CM is assumed to be negligible in comparison to the operating life.

After a PM or CM, the component is restored to *as good as new* condition, and the new cycle of operation starts. Also, the periodic inspection schedule resumes at the same interval, δ and with the same PM threshold, w_P.

The objective of the analysis is to find the optimal δ that would minimize the expected cost or any suitable percentile of the distribution of total cost incurred in the finite time interval $(0,\tau]$. Although the PM threshold w_P can also be considered in the optimization, it is ignored here, because w_P is typically specified by codes and standards.