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Sensitivity study of dynamic systems using polynomial chaos

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ABSTRACT

Global sensitivity has mainly been analyzed in static models, though most physical systems can be described by differential equations. Very few approaches have been proposed for the sensitivity of dynamic models and the only ones are local. Nevertheless, it would be of great interest to consider the entire uncertainty range of parameters since they can vary within large intervals depending on their meaning. Other advantage of global analysis is that the sensitivity indices of a given parameter are evaluated while all the other parameters can be varied. In this way, the relative variability of each parameter is taken into account, revealing any possible interactions. This paper presents the global sensitivity analysis for dynamic models with an original approach based on the polynomial chaos (PC) expansion of the output. The evaluation of the PC expansion of the output is less expensive compared to direct simulations. Moreover, at each time instant, the coefficients of the PC decomposition convey the parameter sensitivity and then a sensitivity function can be obtained. The PC coefficients are determined using non-intrusive methods. The proposed approach is illustrated with some well-known dynamic systems.

1. Introduction

Very often, the equations of a model involve unknown parameters which must be estimated from experimental data. A number of parameters are estimated with more or less precision, which may lead to unacceptable uncertainty on the model output. Among all the parameters, however, only few have a small or insignificant influence on the model response and therefore do not need to be determined precisely. On the other hand, some parameters are decisive for the model response and thus influence its uncertainty significantly. These parameters may require additional measurement data in order to be estimated with relatively high accuracy. To prepare and plan the experiments, it is necessary to distinguish the parameters with an insignificant influence on the response uncertainty, so as to set them at their nominal value in their interval of variation, thanks to the sensitivity analysis. Numerous studies have focused on the sensitivity analysis for static non-linear models, for example [1–7]. The approaches may be local or global. Local approaches help to determine the impact of a small parameter variation around a nominal value [8]. Global approaches also allow the determination of the same impact but by varying the parameter in its entire range of variation. Global methods are often based on the analysis of the output variance and are known as ANOVA (ANalysis Of VAriance) techniques [9,10,7]. More recently, sensitivity moment-independent methods have been used, where emulation model is used to compute density-based sensitivity measure [11]. The emulator is the one of [12].

The model function is split into a sum of functions of increasing dimension [7]. This decomposition, known as High Dimensional Model Representation (HDMR), performs the separation of the effects of different input parameters, which are transmitted in the decomposition of the variance. The present study exclusively focuses on global approaches. In order to quantify the contribution of a parameter to the output variance, a sensitivity index is calculated, often analytically when the model function is known and relatively simple. However, some models may be complex with a high number of parameters so that analytical calculations of the sensitivity indices become time consuming or even impossible. It is therefore necessary to estimate them [3,13–15,7]. Very often, they are computed using Monte Carlo simulations, but for computationally demanding models, this can become intractable. To overcome this drawback, the model of interest is replaced by an analytical approximation, called metamodel, for example, by polynomials which are less expensive. The sensitivity indices are then obtained straightforwardly from the algebraic expression of the coefficients of the polynomial expansion. The polynomial

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chaos (PC) decomposition is an example of such metamodelling. The PC expansion appeared in the 1930s as an effective means to represent stochastic processes in mechanics [16]. It is based on a probabilistic framework and represents amounts from stochastic spectral expansions of orthogonal polynomials [17,18]. It has recently been used in an original manner for sensitivity analysis purposes in [1,19–23]. The great advantage of PC-based sensitivity approaches is that the full randomness of the response is contained in the set of the expansion coefficients.

On the other hand, the analysis of influential parameters is also important for dynamic models since most physical systems (biological, mechanical, electrical and so on) can be described by differential equations. Very few approaches have been proposed in the literature for the sensitivity analysis of dynamic models and the proposed ones are based on local derivatives or on one-at-a-time approaches [24,25]. However, for some applications, mechanical or biological ones for instance, it can be of great importance to consider the entire uncertainty range of parameters since they can vary within large intervals depending on their meaning. Another advantage of global sensitivity analysis is that the sensitivity estimates of individual parameters are evaluated while all the other parameters are varied. In this way, the relative variability of each parameter is taken into account, thus revealing any existing interactions.

The global sensitivity analysis for dynamic models is addressed in this paper. In [26,27], the PC expansion for stochastic differential equations has been studied to represent the model output and to get its statistic properties, but the parameter sensitivity has not been dealt with. Based on these studies, an original approach using the PC decomposition of the output is investigated here, to calculate the parameter sensitivity for dynamic models. At each time instant, the PC coefficients of the decomposition convey the parameter sensitivity and then a sensitivity function of each parameter can be obtained from the algebraic expression of the coefficients. The PC coefficients are determined either by regression or projection techniques which have the advantage of being non-intrusive methods. The proposed approach is illustrated with the well-known mass-spring-damper and DC motor systems.

The outline of this paper is as follows. The sensitivity functions for dynamic systems are presented in Section 2 and the PC expansion for the output of a differential equation in Section 3. Section 4 is focused on the determination of the PC coefficients. Section 5 proposes a PC-based approach to the estimation of the sensitivity functions, which approach is summed up in Section 6. Finally, Section 7 presents an analytical test case to show the convergence of the numerical results. Moreover, the provided approach is applied on some representative dynamic physical systems.

2. Global sensitivity analysis

Consider the following stochastic differential equation:

$$\mathcal{L}(t,\omega,\mathbf{p}(\omega);y)=f(t,\omega,\mathbf{p}(\omega)) \tag{1}$$

where $\mathcal{L}$ is a linear or non-linear differential operator and $\mathbf{p}(\omega)=(p_1(\omega),\ldots,p_n(\omega))$ with $p_i(\omega)$, $i=1, \ldots, n$, the n unknown parameters, considered as uniformly random and independent variables, defined on the unit cube K. The stochastic variable ω is used to indicate the randomness of the input variable $\mathbf{p}$. For the sake of simplicity, ω will be omitted in the following and $\mathbf{p}$ stands for $\mathbf{p}(\omega)$. The solution $y=y(t,\mathbf{p})$, corresponding to the output of the model, can be decomposed into summands of increasing dimension [7], at each time instant:

$$y(t,\mathbf{p})=f_0(t)+\sum_{i=1}^{n}f_i(t,p_i)+\sum_{i=1}^{n-1}\sum_{i<j}^{n}f_{ij}(t,p_i,p_j)+\cdots+f_{1\ldots n}(t,p_1,\ldots,p_n) \tag{2}$$

where $y(t,\mathbf{p})\in\mathbb{R}$ the model output is assumed continuous, derivative and square-integrable. The term $f_0(t)$ is the mean value of the output at each time instant

$$f_0(t)=\int_{K^n}y(t,\mathbf{p})\,\mathrm{d}\mathbf{p} \tag{3}$$

The summands of Eq. (2) are given by

$$\begin{aligned}&f_i(t,p_i)=E[y(t,\mathbf{p})|p_i]-f_0(t)\\&f_{ij}(t,p_i,p_j)=E[y(t,\mathbf{p})|p_i,p_j]-f_i(t)-f_j(t)-f_0(t)\\&\cdots\end{aligned} \tag{4}$$

where $E[y(t,\mathbf{p})|p_i]$ (resp. $E[y(t,\mathbf{p})|p_i,p_j]$) is the conditional expectation of $y(t,\mathbf{p})$ when p_i is set (resp. p_i and p_j are set).

The integral of each summand $f_{i_1,\ldots,i_s}(t,p_{i_1},\ldots,p_{i_s})$ is zero

$$\int_0^1 f_{i_1,\ldots,i_s}(t,p_{i_1},\ldots,p_{i_s})\,\mathrm{d}p_{i_k}=0 \tag{5}$$

with $k\in\{i_1,\ldots,i_s\}$ and $1\le i_1\le\cdots\le i_s\le n$. Due to Eq. (5), the summands are orthogonal to each other

$$\int_{K^n}f_{i_1,\ldots,i_s}(t,p_{i_1},\ldots,p_{i_s})f_{j_1,\ldots,j_r}(t,p_{j_1},\ldots,p_{j_r})\,\mathrm{d}\mathbf{p}=0 \tag{6}$$

for $\{i_1,\ldots,i_s\}\neq\{j_1\ldots j_r\}$.

There are infinite possible decompositions but only one is satisfying Sobol's orthonormality condition (5).

Moreover, the variance of the output, denoted $V(t)$, is given by

$$V(t)=\int_{K^n}(y^2(t,\mathbf{p})-f_0^2(t))\,\mathrm{d}\mathbf{p} \tag{7}$$

The decomposition (2) leads to the following decomposition of the variance $V(t)$:

$$V(t)=\sum_{i=1}^{n}V_i(t)+\sum_{i=1}^{n-1}\sum_{i<j}^{n}V_{ij}(t)+\cdots+V_{1,\ldots,n}(t) \tag{8}$$

with

$$\begin{aligned}&V_i(t)=V[E[y(t,\mathbf{p})|p_i]]\\&V_{ij}(t)=V[E[y(t,\mathbf{p})|p_i,p_j]]-V_i(t)-V_j(t)\\&\vdots\\&V_{1\ldots n}(t)=V(t)-\sum_{i=1}^{n}V_i(t)-\sum_{1\le i<j\le n}V_{ij}(t)-\cdots-\sum_{1\le i_1<\cdots<i_{n-1}\le n}V_{i_1\ldots i_{n-1}}(t)\end{aligned} \tag{9}$$

where $V[E[y(t,\mathbf{p})|p_i]]$ (resp. $V[E[y(t,\mathbf{p})|p_i,p_j]]$) is the variance of the conditional expectation of $y(t,\mathbf{p})$ when p_i is set (resp. p_i and p_j are set).

Since the output $y(t,\mathbf{p})$ varies with time, there is one value for the classic Sobol sensitivity indices at each time instant, thus leading to sensitivity functions. In the same manner as for static models, the sensitivity functions of parameter p_i are obtained by renormalizing (9) with the total variance $V(t)$. Thus, the first order sensitivity function, denoted $S_i(t)$, is defined as follows:

$$S_i(t)=\frac{V_i(t)}{V(t)} \tag{10}$$

The first order sensitivity function $S_i(t)$ represents the main effect of the parameter p_i which corresponds to its contribution alone. The value of $S_i(t)$, at each time instant, lies between 0 and 1. The closer to 1 its value is, the more parameter p_i contributes to the total variance of the output. The sensitivity functions of higher orders, denoted $S_{i_1,\ldots,i_s}(t)$, are defined as

$$S_{i_1,\ldots,i_s}(t)=\frac{V_{i_1,\ldots,i_s}(t)}{V(t)},\quad 1\le i_1\le\cdots\le i_s\le n \tag{11}$$

The sensitivity functions $S_{i_1,\ldots,i_s}(t)$ represent the collective contribution of the parameters $p_{i_1},\ldots,p_{i_s}$, which corresponds to the influence of the interactions of these parameters. By normalizing