Aerospace Science and Technology ••• (••••) •••–•••

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Aerospace Science and Technology

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A variational approach for the dynamics of unsteady point vortices

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ARTICLE INFO

Article history:
Received 26 March 2018
Received in revised form 8 May 2018
Accepted 9 May 2018
Available online xxxx

ABSTRACT

A Lagrangian formulation for the dynamics of unsteady point vortices is introduced and implemented. The proposed Lagrangian is related to previously constructed Lagrangian of point vortices via a gauge-symmetry in the case of vortices of constant strengths; i.e., they yield the exact same dynamics. However, a different dynamics is obtained in the case of unsteady point vortices. The resulting Euler–Lagrange equation derived from the principle of least action exactly matches the Brown–Michael evolution equation for unsteady point vortices, which was derived from a completely different point of view; based on conservation of linear momentum. The proposed Lagrangian allows for applying Galerkin techniques to the weak formulation of the vortex dynamics. The resulting dynamic model of time-varying vortices is applied to the problem of an impulsively started flat plate as well as an accelerating and pitching flat plate. In each case, the resulting lift coefficient using the dynamics of the proposed Lagrangian is compared to that using previously constructed Lagrangian, other models in literature, and experimental data.

1. Introduction

Reduced-order modeling of unsteady aerodynamics has been a topic of research interest since the early formulations of Prandtl [1] and Birnbaum [2]. These formulations were followed by the seminal works of Wagner [3] and Theodorsen [4]; the later efforts of Leishman [5,6] and Peters [7,8]; and in more recent papers by Ansari et al. [9,10], Taha et al. [11] and Yan et al. [12] among others. Because of its ability to account for deforming wakes associated with relatively large amplitude maneuvers, flexible wings, and arbitrary time-varying wing motions, development of the vortex lattice method (UVLM) [13–19] represents a hallmark in the history of unsteady aerodynamic modeling. In DVMs [20–22], a point vortex is released at each time step to satisfy the Kutta condition at the sharp edge it sheds from. Moreover, all of the shed vortices move with constant strengths that have been dictated at the shedding time by the Kutta condition. As such, Helmholtz conservation laws [23] dictate that the dynamics of these constant-strength point vortices will force them to convect with the fluid's local velocity, i.e. the Kirchhoff velocity, see Saffmann [24], pp. 10. Although DVMs were used to develop efficient numerical algorithms to solve for aerodynamic quantities associated with unsteady maneuvers, they require shedding point vortices at each time step, which increases the number of degrees of freedom considerably as the simulation time increases [25,26]. As a remedy, it has been suggested to replace the continuous shedding of constant-strength point vortices [27] with discontinuous/intermittent shedding of varying-strength point vortices, i.e. the strength of the most recent shed point vortex is adjusted each time step to satisfy the Kutta condition, instead of shedding a new vortex to achieve the same objective. Shedding is deactivated until the strength of the unsteady point vortex reaches an extremum [28,29]. At that instant, a new point vortex is shed from the same edge and the previous vortex is convected downstream with the Kirchhoff velocity while keeping its strength constant.

Variational principles have been shown to be useful physical-based approaches for deriving governing equations of both solids and fluids [30,31]. These equations are obtained by setting the first variation of the action, which is the time integral of a candidate Lagrangian function, to zero. Clebsh [30] and Hargreaves [32] derived the equations of motion for an inviscid, incompressible flow by defining the Lagrangian to be the integral of the fluid pressure. Later, Bateman [33] extended the principle to the case of compressible irrotational flow. Luke [34] showed that using variational principles, one is able to provide the boundary conditions by perturbing the limits of integration (Leibniz integral rule). Regarding the vortex motion, Bateman [33], followed by Serrin [35], showed that the equations of motion of vortex lines could be obtained from a variational approach with the ability to regularize

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https://doi.org/10.1016/j.ast.2018.05.010
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Nomenclature

α	flat plate angle of attack	z_k	kth vortex position in Z plane $(x+iy)$
c	flat plate chord	ζ_k	kth vortex position in ζ plane $(\xi+i\eta)$
F_x, F_y	flat plate forces in x and y directions respectively	$\zeta_k^{(I)}$	Image position of kth vortex in ζ plane $(\xi+i\eta)$
Γ_k	kth vortex strength	$\vert\ \vert$	Absolute value of complex number, $\vert z\vert=\sqrt{x^2+y^2}$
w_k	regularized fluid velocity of kth vortex	$\dot{()}$	derivative w.r.t. time
$W, \tilde{W}$	Kirchhoff–Routh functions in flat plate and circle plane respectively	$()^*$	conjugate
z_c	flat plate centroid position in Z plane	$()'$	derivative w.r.t. ζ, $\frac{d}{d\zeta}$

the infinite velocity at the vortex center (Sec. 4 in Ref. [33]). These variational principles were also used to derive governing equations for the cases of fluid motion with distributed vorticity [36] or point vortices [37] with no boundaries, and for the case of a fluid-body interaction [38] that considered constant strength vortices only. Advances made in studying the Hamiltonian dynamics of point vortices [39,37,40] point to the potential of developing a variational principle governing the dynamics of unsteady point vortices interacting with a circular cylinder or a body conformal to it (e.g., airfoil), which is the objective of this work. Such a formulation will allow satisfaction of conservation laws via adding constraints to the variational problem. In addition, it will enable compact and efficient coupling with other variational principles governing rigid body and structural dynamics for coupled unsteady flight dynamics analysis and/or aeroelastic analysis. To date, there have been no developments for variational principles governing the dynamics of unsteady point vortices interacting with solid bodies enclosed by a non-zero total circulation.

The dynamics of constant-strength, point vortices in an inviscid fluid, which is governed by the Biot–Savart law, was derived by Chapman [39] from an action whose Lagrangian is the summation of two functions. The first function is a bilinear function in the vortex spatial coordinates and its velocity, and the second one is the Routh stream function. Recently, Shashikanth et al. [40] proved that the equations of motion for a cylinder moving in the presence of constant-strength vortices of zero sum (i.e., zero total circulation), known as Foppl problem [41,42], have a Hamiltonian structure. Dritschel and Boatto [43] showed similar results for three dimensional differentiable surfaces conformal to a sphere.

In the present work, we present a new Lagrangian function for the dynamics of point vortices that is more general than Chapman's [39]. We examined the relation between the proposed Lagrangian and Chapman's Lagrangian for the cases of constant strength and time-varying point vortices. Interestingly, the proposed Lagrangian dynamics of unsteady point vortices recovers the momentum based Brown–Michael model [44]. We applied the Galerkin technique to the resulting weak formulation of the time-varying vortices for the problem of an impulsively started flat plate as well as an accelerating and pitching flat plate, with comparison to experimental data in the literature [45,46]. To the best of our knowledge, this is the first variational principle to govern the dynamics of unsteady point vortices.

2. Lagrangian dynamics of point vortices

2.1. General formulation

Considering the flow around a sharp-edged body (in the z-plane) and mapping it to the flow over a cylinder (in the ζ-plane) with an interrelating conformal mapping $z=z(\zeta)$, as shown in Fig. 1, the regularized local fluid velocity (Kirchhoff velocity) of the shed kth vortex is given by [47–49]

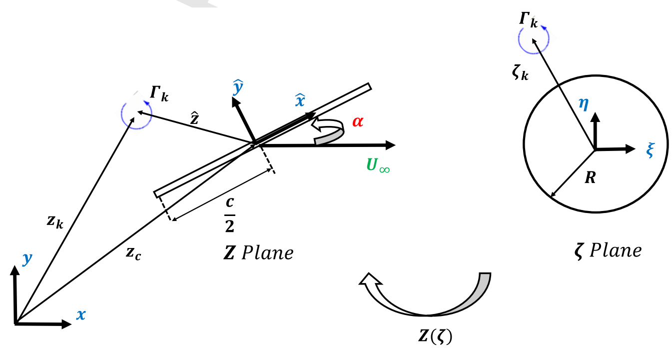

Fig. 1. Conformal mapping between a sharp-edged body and a circular cylinder.

$$\begin{aligned}\frac{dz_k}{dt} &= w_k(z_k)\\ &= \frac{1}{[z'(\zeta_k)]^*}\lim_{\zeta\longrightarrow\zeta_k}\left[\frac{\partial F}{\partial \zeta}-\frac{\Gamma_k}{2\pi i}\frac{1}{\zeta-\zeta_k}-\frac{\Gamma_k}{4\pi i}\frac{z''(\zeta)}{z'(\zeta)}\right]^*\end{aligned} \tag{1}$$

where F is the complex potential, Γ_k is the strength of the kth vortex, and the asterisk refers to a complex conjugate. The last term on the right hand side, which involves the second derivative of the transformation, was first derived by Routh then by Lin [47] and later by Clements [48].

Lin [50] showed the existence of a Kirchhoff–Routh function W (Ref. [51] sec. 13.48) that relates the velocity components of the kth vortex to the derivatives of W, in a Hamiltonian form such that the velocity components of the vortex in z plane are

$$\begin{aligned}\Gamma_k u_k &= \frac{\partial W}{\partial y_k}\\ \Gamma_k v_k &= -\frac{\partial W}{\partial x_k}\end{aligned} \tag{2}$$

The Kirchhoff–Routh function $\tilde{W}$ in the circle plane is related to the stream function ψ_0 by [47,51,52]

$$\begin{aligned}\tilde{W}(\xi_k,\eta_k) &= \Gamma_k\psi_o(\xi_k,\eta_k)\\ &\quad+\sum_{k,l,k\neq l}\frac{\Gamma_k\Gamma_l}{4\pi}\left[\ln|\zeta_k-\zeta_l|-\ln|\zeta_k-\zeta_l^I|\right]\\ &\quad+\sum_k\frac{\Gamma_k^2}{4\pi}\ln|\zeta_k-\zeta_k^{(I)}|\end{aligned} \tag{3}$$

where ψ_o is the stream function of the body motion (i.e., $F=F_0+\sum_{k=1}^n\Gamma_k$ and $F_0=\phi_0+i\psi_0$). Then the relation between the Kirchhoff–Routh function W in the flat plate plane and that in circle plane $\tilde{W}$ is given as [47]:

$$W=\tilde{W}+\sum_k\frac{\Gamma_k^2}{4\pi}\ln|\frac{dz}{d\zeta}| \tag{4}$$

Please cite this article in press as: A.A. Hussein et al., A variational approach for the dynamics of unsteady point vortices, Aerosp. Sci. Technol. (2018), https://doi.org/10.1016/j.ast.2018.05.010