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Short communication

Optimal nozzle Mach number for maximizing altitude of sounding rocket

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ABSTRACT

A pseudo-analytic approach is applied to determine the optimal nozzle Mach number for maximizing the altitude of a sounding rocket flying in a standard atmosphere. The one-dimensional rocket momentum equation including thrust, gravitational force and aerodynamic drag is considered, for which it is impossible to obtain an analytic solution in a general form. In this work, a piecewise pseudo-analytic approach with a constant parameter introduced to make the velocity integral in the governing equation analytic is applied. The rocket flight in the standard atmosphere is analyzed by dividing the entire range into small intervals where the drag parameter and the gravitational acceleration can be treated as a constant in each interval. A characteristic equation exists and provides accurate predictions of the optimal nozzle Mach number for maximizing the altitude at burn-out state or at apogee.

1. Introduction

In most cases, the design target of a sounding rocket is the altitude at burn-out state or at apogee. The rocket altitude can change according to the ejection conditions of the propellant jet through a supersonic nozzle. Therefore, it is necessary to determine an optimal thrust condition for maximizing the altitude for a given launching condition, known as the Goddard problem, which has been extensively studied using variation methods, asymptotic approaches, and optimal control theories [1–4]. A previous study [5] presented an analytic approach to determine the optimal conditions for the typical situations where an analytic solution exists. This approach was extended to rocket flight in a standard atmosphere [6] by applying the divide-and-conquer strategy. However, the existence of an analytic solution requires the balance of the three forces and thus requires control of the thrust. Most sounding rockets use a constant mass flow rate of propellant through a fixed nozzle in which the rocket motion cannot be solved with an analytic approach. Thus, these analytic approaches have serious limitations in real applications. A successive study [7], also exploiting the divide-and-conquer strategy, presented a pseudo-analytic approach to overcome this limitation and showed that the optimal mass flow rate of a rocket could be determined. However the nozzle Mach number that is another key parameter for the optimal thrust condition was not considered.

Hence in the present study an effective method will be established to determine the optimal nozzle Mach number of a sounding rocket. The present study extends the pseudo-analytic concepts and methods of the previous studies [6,7] in order to determine the optimal nozzle Mach number for the maximum altitude at burn-out state or at apogee. It is difficult to determine the optimal mass flow rate and the optimal nozzle Mach number simultaneously, since the two key parameters are highly coupled in the thrust function. Hence, in the present study, an iterative approach is used. First, the optimal mass flow rate is obtained with a fixed nozzle Mach number, and then the optimal nozzle Mach number is calculated with the fixed optimal mass flow. The present methods will give a way to determine the full parameters for the optimal thrust condition and thus valuable information for an effective design of a sounding rocket.

The motion of a sounding rocket launched in the vertical direction is considered for simplicity. Then, the motion of a sounding rocket can be described using a one-dimensional momentum equation that includes thrust, gravitational force, and aerodynamic drag. The rocket model considered in the present study is the same one used in the previous study [6,7] and is a simplified model based on the Korea Sounding Rocket Program (KSR II and III) [8]. The air density is calculated based on the US standard atmosphere [9] and the variable gravitational acceleration changing with altitude is considered.

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Nomenclature

A	Area	m^2
F	thrust	N
g	gravitational acceleration	m/s^2
h	altitude	m
J	pseudo drag parameter	
K	drag parameter	
M	Mach number	
m	rocket mass	kg
$\dot{m}$	rate of rocket mass change or mass flow rate of propellant jet	kg/s
q	velocity parameter for rocket velocity	m/s
r	control parameter for rocket velocity	m/s
u_e	exhaust velocity	m/s
v	rocket velocity	m/s
p	static pressure	Pa
T	temperature	K
x	non-dimensional scale to express rocket velocity	
β	coefficient of derivative of mass with respect to control parameter	
γ	specific heats ratio	
φ	parameter to express derivative of pseudo drag parameter	
ρ	density	kg/m^3
ψ	parameter to express intermediate state between burn-out state and apogee	
Ω	rocket mass ratio between total mass and dry mass	
ω	rocket mass ratio between adjacent intervals	

Subscripts

a	ambient air
b	burn-out state
e	exhaust condition at rocket nozzle exit
i	index of an intermediate divided interval
k	index of iteration step
n	index of final divided interval
o	ground state
s	stationary state (apogee)
$temp$	temporary value
$*$	matching at burn-out state

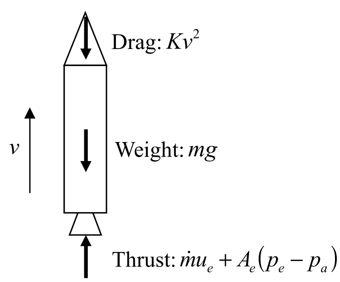

Fig. 1. Schematic of a rocket and applied forces.

2. One-dimensional rocket momentum equation

2.1. Governing equation in boost phase

The one dimensional motion of a rocket climbing in the vertical direction is considered in the present study. Fig. 1 shows the schematic of the one-dimensional rocket motion and applied forces.

The motion of a rocket in boost phase can be described with the following one-dimensional rocket momentum equation including thrust, gravitational force and aerodynamic drag as follows [10–12]:

$$m\frac{dv}{dt} = F - mg - Kv^2, \tag{2.1a}$$
$$F = \dot{m}u_e + A_e(p_e - p_a), \tag{2.1b}$$
$$K = \frac{S}{2}C_d\rho_a. \tag{2.1c}$$

The rocket mass decreases with the mass flow rate of propellant. The mass flow rate is equal to the rate of change of the rocket mass and has a negative sign. The exhaust velocity u_e has a negative sign, since its direction is opposite to the rocket velocity; thus, the velocity thrust term $\dot{m}u_e$ has a positive sign.

In case the thrust condition is fixed, we cannot obtain an analytic solution in a general form due to the nonlinear behavior of the governing equation. The previous study [7] suggested a pseudo-analytic approach to avoid such difficulties, such as extending the "divide-and-conquer" strategy. The governing equation can then be represented as follows:

$$\frac{dv}{r^2 - v^2} = \frac{J}{\dot{m}}\frac{dm}{m}, \tag{2.2a}$$
$$J = K\frac{q^2 - v^2}{r^2 - v^2}, \tag{2.2b}$$
$$q = \sqrt{\frac{F - mg}{K}} = \sqrt{\frac{\dot{m}u_e + A_e(p_e - p_a) - mg}{K}}, \tag{2.2c}$$
$$r = \sqrt{\frac{\dot{m}u_e}{K_*}}, \tag{2.2d}$$
$$K_* = K_b\left(1 + \frac{A_e(p_e - p_{a,b}) - m_b g_b}{\dot{m}u_e}\right)^{-1}. \tag{2.2e}$$

The piecewise pseudo-analytic solution of the governing equation at the state (n) is expressed in the following form.

$$v_n = r\frac{x_n - 1}{x_n + 1}, \tag{2.3a}$$
$$x_n = \exp\left[-\frac{2u_e}{K_* r}\sum_i^n \bar{J}_i \ln(\omega_i)\right] = \exp\left[-\frac{2r}{\dot{m}}\sum_i^n \bar{J}_i \ln(\omega_i)\right], \tag{2.3b}$$
$$\omega_n = \frac{m_{n-1}}{m_n} > 1. \tag{2.3c}$$

The altitude of a rocket at burn-out state can be obtained using the time integration of the velocity as follows:

$$h_b = \int_o^{t_b} v\,dt = \frac{1}{\dot{m}}\sum_{i=1}^{b}\int_{m_{i-1}}^{m_i} r\frac{x-1}{x+1}dm. \tag{2.4}$$

The above integral cannot be solved analytically, and thus should be calculated numerically. In the present study, numerical integration using Simpson's rule [14] is applied.

2.2. Governing equation in coast phase

After the propellant of the rocket is totally consumed, the flight phase turns into coast phase, where the rocket climbs under its own decreasing inertial speed until the stationary state or apogee. The rocket momentum equation then becomes as follows: