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Modelling imperfect maintenance and the reliability of complex systems using superposed renewal processes

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ABSTRACT

In problems of maintenance optimization, it is convenient to assume that repairs are equivalent to replacements and that systems or objects are, therefore, brought back into an as good as new state after each repair. Standard results in renewal theory may then be applied for determining optimal maintenance policies. In practice, there are many situations in which this assumption cannot be made. The quintessential problem with imperfect maintenance is how to model it. In many cases it is very difficult to assess by how much a partial repair will improve the condition of a system or object and it is equally difficult to assess how such a repair influences the rate of deterioration. In this paper, a superposition of renewal process is used to model the effect of imperfect maintenance. It constitutes a different modelling approach than the more common use of a virtual age process.

1. Introduction

The research presented in this paper was motivated by the practice of spot repairing steel coating systems (i.e. paint). As soon as some percentage of the coating is damaged, the damaged spots are repaired by replacing the old coating. Since not all coating is replaced at once, these spot repairs represent partial (or imperfect) repairs of the coating system.

Let us consider the following example: we have a steel structure which is protected against corrosion by a coat of paint. Once 3% of the total surface of the coating is damaged, these damaged areas are repaired. The remaining 97% of the surface is left as such. From a modelling point of view the difficulty now lies in the fact that the (random) time to reach the threshold of 3% damaged surface is different after the repair than it was before. Probabilistically we now have a mixture of two areas with different lifetime expectations. After the second repair, the mixture will become more complicated as we may repair some spots with the old coating, but also some of the spots which received a new layer at the first repair.

With imperfect maintenance, the rate of deterioration, therefore, changes after every maintenance action and it is this behaviour which is difficult to model. Consider for example, the lifetime-extending maintenance (LEM) module introduced in [1] and discussed in [2]. It is a stand-alone software program that can be used to assess whether a combination of lifetime-extending maintenance and preventive replacements is more economical compared to a policy with only preventive or corrective replacements. The user must select one of two options for the rate of deterioration after imperfect maintenance: (1) the rate is the same as at the start of the service life or (2) the rate is the same as right before the maintenance is performed. Fig. 1 shows the difference between these two options. The first option may be considered as a kind of "as good as new" situation and the second option as a kind of "as bad as old" situation as far as the rate of deterioration is concerned. Allowing only these two options makes it easier to implement such a module, but in reality the rate of deterioration after imperfect maintenance will be somewhere in between these two extremes.

Intuitively, the time between repairs will converge to some value as the number of spot repairs increases. This may be easily demonstrated by means of a Monte Carlo simulation as will be done in the examples later on. In the long run the ages of all spots become sufficiently mixed for some form of stationarity to arise. In this paper, the full probability distribution function of the time between repairs will be derived.

The basic idea in this paper is the following: the coated surface consists of a grid of cells which represent the spots. If we assume that the arrival in time of damages to the coating in each cell can be modelled by a renewal process, then the arrival of damages on the complete surface is formed by a superposition of these processes. For simplicity we assume that the individual cells are independent. Implicitly we have also assumed that damaged cells are repaired immediately and that the time required for the repair is negligible.

However, in this case, we do not want to repair the cells once they become damaged, but we want to wait until a fraction of the

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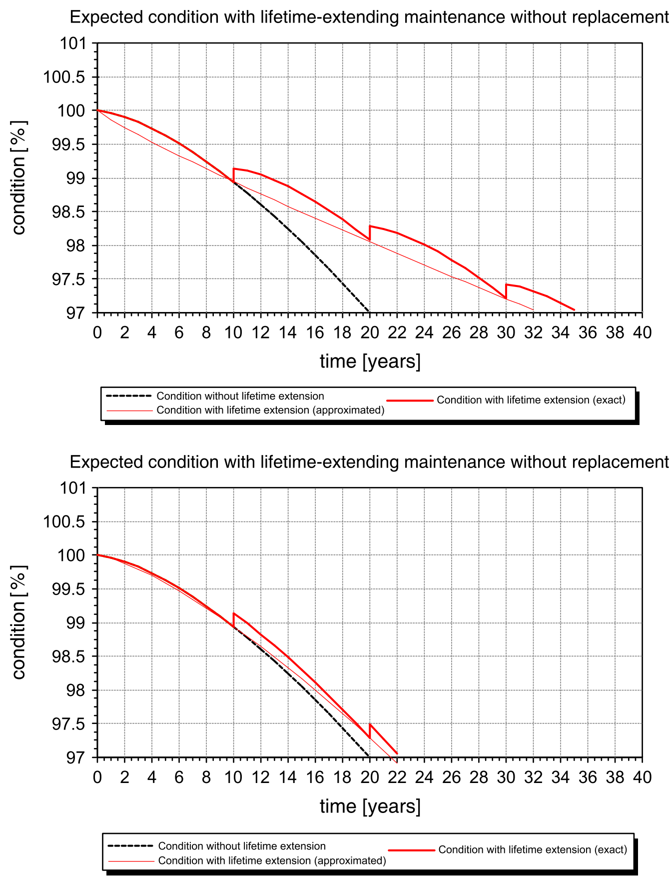

Fig. 1. Two options for the rate of deterioration after imperfect maintenance in the LEM module: same as at start of service life (top) or same as before the last maintenance action (bottom).

total number of cells is damaged and then repair all of the damaged cells at once. Using the concept of the superposed renewal processes, we will approximate the asymptotic distribution of the time between repairs or, more general, the time to reach a predefined fraction of damaged cells.

Another example of a similar situation is one of a parallel system consisting of a large number of identical components. A certain fraction of the total number of components is allowed to fail before they are collectively replaced. This fraction could be based on the requirement of a minimum system reliability or simply on the fact that the stress on the remaining components will become too large at some point in time.

Section 2 discusses two approaches to derive the probability distribution function of the interarrival time for a superposed renewal process. Then, Section 3 uses this result to determine the probability distribution function of the time between repairs. These results are demonstrated in Section 4 using a number of examples. The paper ends with a summary and conclusions.

2. Superposition of renewal process

We start by introducing the notation and by giving the necessary definitions. A renewal process $\{N(t), t \geq 0\}$ is a non-negative integer-valued stochastic process which registers the successive renewals in the time interval $[0,t]$. Let each renewal process have interarrival times X_1, $X_2, \ldots$ and let $S_k = \sum_{i=1}^{k} X_i$, $k \geq 1$, be the time of the k-th renewal such that $N(t) = \sum_{k=1}^{\infty} 1_{S_k \leq t}$. Each interarrival time is identically and independently distributed according to some non-negative probability distribution function $F(x) = \Pr\{X_i \leq x\}$, $x \geq 0$. The hazard rate is given by the familiar expression $v(x) = f(x)/R(x)$, where $R(x) = 1 - F(x)$ is the survival probability at a time x since the last renewal. The following well-known relationships will also be required later on

$$R(x) = \exp\left\{-\int_{u=0}^{x} v(u)\,du\right\} \tag{1}$$

and

$$f(x) = v(x)\exp\left\{-\int_{u=0}^{x} v(u)\,du\right\}. \tag{2}$$

The rate of occurrence (or the average process rate) is $1/\mu$, where

$$\mu = E[X_i] = \int_{x=0}^{\infty} x\,dF(x) \tag{3}$$

is the mean time between renewals. For more on the theory of renewal processes, we refer to any textbook on stochastic processes like, for example, [3].

A superposed renewal process, here denoted by $N_s(t)$, is obtained by counting the renewals up to time t of multiple source processes, each of which is a renewal process:

$$N_s(t) = \sum_{i=1}^{n} N_i(t), \quad n \geq 1.$$

Unless the individual renewal processes $N_i(t)$ are Poisson processes, the superposed process itself is not a renewal process. This is because the interarrival times are no longer independent. However, the distribution of the interarrival time of the superposed process can be derived analytically. Moreover, it is well known that the superposed process is close to being a Poisson process when the number of sources is very large (see Section 2.3 for a discussion of this property).

In the following, we will discuss two distinctively different approaches to the derivation of the distribution of the interarrival time of the superposed process. One is the approach by [4] using a rate-optimal approximation of the superposed process and the other is the approach by [5] using what is known as the equilibrium distribution of a renewal process. Given the same assumptions about the individual renewal processes, we will show that these two approaches yield the same result. In the next section, we will show how this result can be used to obtain an approximation of the inter-repair time distribution. Any (probability distribution) function associated with the superposed process will be denoted with a subscript s. We begin with the approach used by [5].

2.1. Cox and Smith (1954)

First, we give a brief account of the main contribution in the paper. Let $Y(t)$ denote the age of the process at time t: $Y(t) = t - S_{N(t)}$, with $S_{N(t)}$ the time of the last renewal before t. Cox and Smith [5] referred to this as the delay time. If the interarrival time distribution $F(x)$ is not lattice, then

$$\lim_{t\to\infty} \Pr\{Y(t) \leq x\} = \frac{1}{\mu}\int_{u=0}^{x} R(u)\,du. \tag{4}$$

This limit distribution is referred to as the equilibrium distribution for F, see Ross [6, Chapter 3], and we will denote it by $G(x)$. For a superposed process, $Y_s(t)$ is equal to the smallest age of the individual source processes: $Y_s(t) = \min\{Y_1(t), \ldots, Y_n(t)\}$. The distribution can thus be easily derived as follows:

$$\Pr\{Y(t) > x\} = \prod_{i=1}^{n} \Pr\{Y_i(t) > x\} \tag{5}$$

and if we assume that t is large or, in other words, that the process has been running for a long time

$$\Pr\{Y(t) > x\} = [1 - G(x)]^n. \tag{6}$$