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Bayesian Monte Carlo method

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ABSTRACT

To reduce cost of Monte Carlo (MC) simulations for time-consuming processes, Bayesian Monte Carlo (BMC) is introduced in this paper. The BMC method reduces number of realizations in MC according to the desired accuracy level. BMC also provides a possibility of considering more priors. In other words, different priors can be integrated into one model by using BMC to further reduce cost of simulations. This study suggests speeding up the simulation process by considering the logical dependence of neighboring points as prior information. This information is used in the BMC method to produce a predictive tool through the simulation process. The general methodology and algorithm of BMC method are presented in this paper. The BMC method is applied to the simplified break water model as well as the finite element model of 17th Street Canal in New Orleans, and the results are compared with the MC and Dynamic Bounds methods.

1. Introduction

In the framework of determining reliability of complex systems, we will focus our attention in this paper on the Monte Carlo methods. This is necessary because Monte Carlo methods are flexible and more suited for dealing with complicated problems. The Monte Carlo family of methods includes the so-called variance optimization schemes for improving the computational efficiency. Among these the most widely used are the importance sampling (IS), explained in [5], directional sampling (DS), presented in [6] and the extended methods addressed by [17]. Each method has its advantages and disadvantages when it is coupled with complex problems. These existing methods have been discussed in greater details in [7, Chapter 2].

In general, which Monte Carlo simulation is widely considered to be among the most robust and most generally applicable method for the reliability analysis of engineering structures. The Monte Carlo technique considers each simulation independent of the previous simulations. The absence of systematic errors and the fact that its error analysis is well-understood are properties that many competing methods lack. A drawback, however, is the often large number of runs needed, particularly in complex models, where each run may entail a finite element analysis or other time consuming procedure. Variance reduction methods may be applied to reduce simulation cost as was reviewed above. In this paper, we describe a method to reduce the simulation cost, while retaining the accuracy of Monte Carlo. Here, we present a method to speed up the simulation process by producing a predictive and judgmental tool. Having this tool, we will be able to predict the response of model or conclude with certainty that if the sample is to be accepted or rejected. We achieve this by implementing some logical dependence among neighboring pixels (points). This is obtained considering the fact that each randomly generated point may be estimated by its neighboring pixels (points). This implementation is done by using the Bayesian technique which is, in fact, based on the Bayesian interpolation in [1]. As a result, we assign a probability density function (PDF) to an arbitrary or randomly generated point. Therefore, given the desired level of tolerance, we judge the accuracy of prediction for this point. As a result, the information of the neighborhoods are implemented into the model. In this technique, there is still possibility of considering more priors to the model that can be used for different models. For instance, one may be interested in integrating the monotonic behavior of flood defence structures as presented in [9] to further reduce calculation efforts by using the BMC method.

This study requires a background of Bayesian techniques as well as Monte Carlo methods. The Bayesian approach is well described in [3]. An instructive reference for applying Bayes' Theorem into practice is [12]. The Monte Carlo method in details is presented for instance in [5,2].

The layout of this paper as follows. Section 2 presents an overview of the BMC method. The prior, likelihood, and posterior of the BMC method are extensively described in Sections 3–5, respectively. Further information on the mathematical formula and methodology is presented in Section 5.1. The BMC method and its algorithm are described in Section 7. Numerical examples are presented in Section 8. Conclusions are presented in Section 9.

2. Overview

To start modeling the problem, we assume that there is a signal U which is to be estimated at a number of discrete points. These discrete points will be called pixels, presented by u_i. These

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pixels present a vector of pixels $\mathbf{u} \equiv (u_0, \ldots, u_{v+1})$. Therefore, there are totally $v+2$ pixels. The first and last pixels, presented by u_0 and u_{v+1}, are called boundary pixels and are treated separately. As a result, v presents the number of interior pixels. The total number of observed data points is equal to n which are distributed in arbitrary (or random) locations among the pixels. Therefore, the maximum value of n is equal to $v+2$ when there is an observed data point for each pixel ($n \le v+2$). Locations of the observed data points are collected in vector $\mathbf{c}$ that has n elements presented by c_i, where $i=1,2,\ldots,n$. The vector of observed data points is called $\mathbf{d} \equiv (d_1, \ldots, d_n)$, and its elements are presented by d_i. Fig. 1 presents an illustration of the internal and boundary pixels as well as data points. According to this figure, for instance, $\mathbf{c} \equiv (1, v-1, v+2)$.

The univariate probability density function (PDF) for an arbitrary pixel, given the data D and the informational context I, will be found by integrating out all pixels. In this case the sum rule is applied and the product is integrated all over the multivariate posterior PDF of all pixels of U except the required pixel u_j:

$$P(u_j|D,I) = \int P(U|D,I) \underbrace{\ldots du_i \ldots}_{i \neq j}. \tag{1}$$

Also, according to the Bayes' rule we have

$$P(U|D,I) = \frac{P(D|U,I)P(U|I)}{P(D|I)}, \tag{2}$$

where $P(D|I)$ is a normalization constant. Therefore, combination of Eqs. (1) and (2) produces the following equation:

$$P(u_j|D,I) \propto \int P(D|U,I)P(U|I) \underbrace{\ldots du_i \ldots}_{i \neq j}. \tag{3}$$

This equation presents needs to define $P(D|U,I)$ and $P(U|I)$ which are called the likelihood and prior, respectively. The likelihood, or in this case more appropriate the PDF of the data (D) conditional on the pixels (U), is constructed by making the standard assumptions of noise. It is assumed that this noise has a zero mean value and a specific standard deviation. As a result, it is important to define the prior on the base of the available prior information.

3. The prior

There are some logical dependence among neighboring pixels and this expectation is translated in the following model, f, for an arbitrary pixel u_i. In this model, the value of u_i is estimated by its two neighbor points. Fig. 2 clarifies this concept where two neighbor points of pixel i are shown, and their positions are shown by x. Given an arbitrary location in $[x_{i-1}, x_{i+1}]$, it is logical to assume that a closer neighbor has more influence on the estimate than the other neighbor.

Therefore, $\delta_{r,i}$ and $\delta_{l,i}$ are two relative weights which, respectively, present the influence of left and right neighbor points on the target pixel, and one gets $\delta_{l,i}+\delta_{r,i}=1$. As a result, the value of the pixel u_i is estimated as

$$\hat{u}_i = f(u_{i-1}, u_{i+1}) = u_{i-1} \cdot \delta_{r,i} + u_{i+1} \cdot \delta_{l,i}, \tag{4}$$

where $\delta_{r,i}$ and $\delta_{l,i}$ are the absolute distance from the right and left points, respectively. Therefore, one obtains

$$\delta_{l,i} = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \quad \text{and} \quad \delta_{r,i} = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}. \tag{5}$$

The same concept holds to higher dimensions. Having two vectors $\mathbf{x}_{i-1}$ and $\mathbf{x}_{i+1}$ that are produced from three successive points ordered according to a certain axis, j, we will have

$$\delta_{l,i}^{(j)} = \frac{|\mathbf{x}_{i-1}|}{|\mathbf{x}_{i-1}| + |\mathbf{x}_{i+1}|} \quad \text{and} \quad \delta_{r,i}^{(j)} = \frac{|\mathbf{x}_{i+1}|}{|\mathbf{x}_{i-1}| + |\mathbf{x}_{i+1}|}, \tag{6}$$

where $|\mathbf{x}|$ is the length of vector $\mathbf{x}$ calculated as

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}.$$

For instance, a two-dimensional problem is shown in Fig. 3. This figure presents three arbitrary pixels in which data points are assigned to two of them. In this case, the relative weights that present the influence of left and right neighbors are shown for the x axis as

$$\delta_{l,i}^{(x)} = \frac{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} + \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}}, \tag{7}$$

$$\delta_{r,i}^{(x)} = \frac{\sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} + \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}}, \tag{8}$$

and the same equations hold for the y axis. Having the model defined, the error e_i also is implicitly defined by

$$e_i = u_i - \hat{u}_i = u_i - f(u_{i-1}, u_{i+1}) = u_i - (u_{i-1} \cdot \delta_{r,i} + u_{i+1} \cdot \delta_{l,i}). \tag{9}$$

The only thing we know about this error is that the error has a mean of zero (the error is either positive or negative) with some unknown local variance ϕ_i^2. It means that we assume that the standard deviation of error is relevant to the distance of neighbors, $\phi_i \propto (x_{i+1} - x_{i-1})$. As a matter of fact, a closer neighbor points to the pixel a smaller error for the estimation is expected. Using the principle of Maximum Entropy [3], we find the well known Gaussian probability density function of e_i presented in Eq. (10). ϕ_i in this equation stands for the standard deviation of the pixel u_i:

$$P(e_i|\phi_i) = \frac{1}{\sqrt{2\pi}\phi_i} \exp\left[-\frac{1}{2\phi_i^2} e_i^2\right]. \tag{10}$$

Substituting Eq. (9) into Eq. (10) and making the appropriate change of variable from e_i to u_i, the PDF of pixel u_i can be obtained by

$$P(u_i|u_{i-1}, u_{i+1}, \phi_i, I) = \frac{1}{\sqrt{2\pi}\phi_i} \exp\left[-\frac{1}{2\phi_i^2} [u_i - u_{i-1} \cdot \delta_{r,i} - u_{i+1} \cdot \delta_{l,i}]^2\right]. \tag{11}$$

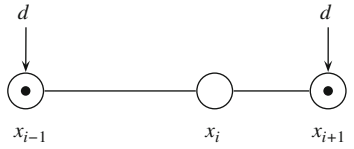

Fig. 2. One-dimensional illustration of the pixels which data points are assigned to.

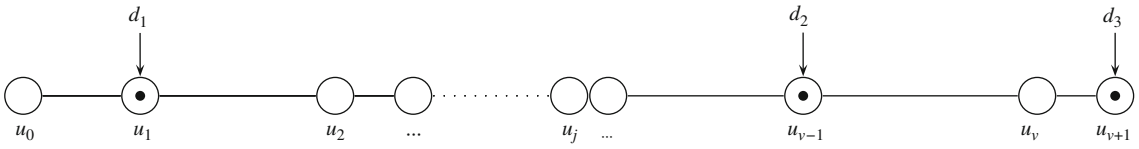

Fig. 1. A one-dimensional illustration of the pixels which data points are assigned to.