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Estimation of a launch vehicle stage fallout zone with parametric and non-parametric importance sampling algorithms in presence of uncertain input distributions

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ABSTRACT

The estimation of launch vehicle fall back safety zone is a crucial problem in space application since the consequence of a mistake may be dramatic for the population. It consists in estimating the probability that the distance between the launcher stage fall-back position calculated, with a trajectory simulation code, and the predicted one is lower than a given threshold. This probability of having a distance that exceeds the critical limit is of course low and may hardly be estimated with crude Monte Carlo methods. One proposes in this paper adaptive importance sampling algorithms when some parameters of input probability density suffer from uncertainty. This situation is a difficult issue for computational reasons since a complete importance sampling procedure is necessary to estimate the rare event probability associated to each combination of input density parameters. It is no more the case with the proposed parametric and non-parametric approaches based on the definition of a more adapted initial sampling density.

1. Introduction

During the launch of a satellite, the most important event is of course the lift-off. Nevertheless, a successful launch is not the end of the launcher task. Once their mission is completed, the launch vehicle stages are jettisoned and fall back to land or in the ocean. The estimation of launch vehicle fallout safety zone is a crucial problem in aerospace because it potentially involves dramatic repercussions on the population and the environment. For that purpose, an efficient estimation of the probability that a launch vehicle stage falls at a distance greater than a given safety limit is strategic for the qualification of such vehicles. The fact that a launcher stage touches the ground far from its intended point is a rare event and its probability is consequently difficult to estimate with crude Monte Carlo simulations.

Rare event probability estimation is a growing part of complex system simulation for safety and risk analysis. Numerous statistical and simulation techniques have been proposed to estimate such probabilities. Indeed, importance sampling [1–3], subset simulation [4,5], first and second order reliability methods [6,7], or extreme value theory [8,9] are notably well known algorithms to estimate rare event probabilities on input–output "black-box" functions such as aerospace simulation codes. Their principles and advantages/drawbacks have also been deeply studied [10]. In this article, we will focus more precisely on importance sampling techniques, with parametric or non-parametric optimization.

In order to perform the rare event probability estimation, several parameters $\boldsymbol{\Theta}$ of the simulation are set, such as the different parameters of input parametric density, and influence the probability estimate. A general approach consists in estimating the rare event probability associated to each combination of these different parameters with a complete importance sampling procedure. Nevertheless, this method can be computationally cumbersome when the simulation of the complex system is time consuming. To solve this problem, we propose to use the samples obtained to estimate the rare event probability when the input distribution parameters are fixed to a reference value $\boldsymbol{\Theta} = \boldsymbol{\theta}$ as an initialization of the importance sampling procedure adapted to a new value of density parameter $\boldsymbol{\Theta} = \boldsymbol{\theta}'$.

This paper is organized as follows. We first briefly review the principles of parametric and non-parametric importance sampling algorithms. Then, we describe the proposed adaptive sampling approaches to handle uncertain input distribution parameters. The

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final section is dedicated to the application of the proposed algorithms on different analytical and aerospace test cases in order to assess their efficiency with respect to classical methods.

2. Quick review on adaptive importance sampling techniques

In this first section, we briefly review some basics on rare event estimation and more precisely on importance sampling. For the sake of conciseness, the complete details of the following techniques are not given in this article. The reader can refer to [10] for more information.

2.1. Context

Let us define a d-dimensional continuous random vector $\mathbf{X}$ with a probability density function (pdf) $h_0(\cdot)$. In this article, we focus on the estimation of the probability that $P(\phi(\mathbf{X}) > S)$ with $\phi(\cdot)$, a continuous input–output scalar function $\phi(\cdot) : \mathbb{R}^d \to \mathbb{R}$ and S a threshold. The launch vehicle stage fall-back use case may be modeled as an input–output function $\phi(\cdot)$ with as inputs several launch vehicle characteristics and environmental conditions, and as output the distance between the estimated and the predicted fall-back positions. In this article, $\phi(\cdot)$ is a complex simulation code developed at ONERA [11].

We assume here that $Y = \phi(\mathbf{X})$ is a random variable. Crude Monte-Carlo simulations [12] are a simple approach to estimate the probability $P(\phi(\mathbf{X}) > S)$ but they are in fact not well suited to rare event probability estimation. Different alternatives to Monte-Carlo can be considered, such as importance sampling [1–3], importance splitting [4,5] or extreme value theory [8,9]. Only importance sampling techniques are analyzed in this article.

2.2. Adaptive importance sampling

2.2.1. Principle of importance sampling

The objective of importance sampling (IS) is to reduce the variance of the Monte-Carlo probability estimator without increasing the number of required simulations N. The main idea is to generate the multidimensional samples $\mathbf{X}_1, \ldots, \mathbf{X}_N$ with an auxiliary density $h(\cdot)$, which is able to generate more samples such that $\phi(\mathbf{X}) > S$ than $h_0(\cdot)$, and then to introduce a weight in the probability estimate [10]. The IS probability estimate $\widehat{P}^{IS}$ is then given by

$$\widehat{P}^{IS} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\phi(\mathbf{X}_i)>S} \frac{h_0(\mathbf{X}_i)}{h(\mathbf{X}_i)}, \tag{1}$$

where $\mathbf{1}_{\phi(\mathbf{X}_i)>S}$ is equal to 1 if $\phi(\mathbf{X}_i) > S$ and 0 otherwise. The optimal auxiliary density $h_{opt}(\cdot)$ (*i.e.* minimizing the estimator variance) is given in [13]

$$h_{opt}(\cdot) = \frac{\mathbf{1}_{\phi(\cdot)>S} h_0(\cdot)}{P}. \tag{2}$$

Since the optimal auxiliary density $h_{opt}(\cdot)$ depends unfortunately on the probability of interest P, it cannot be determined in advance. Two main algorithms have been proposed to determine an efficient sampling pdf.

2.2.2. Cross-entropy optimization of importance sampling auxiliary density

Let us define $h_\lambda(\cdot)$, a family of densities indexed by a parameter $\lambda \in \Delta$ where Δ is the multidimensional space of pdf parameters. The parameter λ is, for instance, the mean and the covariance matrix in the case of Gaussian densities. The objective of importance sampling with cross-entropy (CE) is to determine the parameter λ_{opt} that minimizes the Kullback–Leibler divergence between $h_\lambda(\cdot)$ and $h_{opt}(\cdot)$. This process is adaptive since the direct determination of λ_{opt} is difficult. In fact, one proceeds iteratively to estimate λ_{opt} with an iterative sequence of thresholds,

$$q_0 < q_1 < q_2 < \cdots < q_k < \cdots < S$$

chosen adaptively using quantile definition. The complete details of this algorithm can be found in [14,15]. The cross-entropy optimization algorithm for importance sampling density is given by

(i) $k = 1$, define $h_{\lambda_0}(\cdot) = h_0(\cdot)$ and set $\rho \in (0, 1)$
(ii) Generate the population $\mathbf{X}_1^{(k)}, \ldots, \mathbf{X}_N^{(k)}$ according to the pdf $h_{\lambda_{k-1}}(\cdot)$ and apply the function $\phi(\cdot)$ in order to have $Y_1 = \phi(\mathbf{X}_1^{(k)}), \ldots, Y_N = \phi(\mathbf{X}_N^{(k)})$
(iii) Compute the empirical ρ-quantile $q_k = Y_{\lfloor \rho N \rfloor}$, where $\lfloor a \rfloor$ denotes the largest integer that is smaller than or equal to a
(iv) Optimize the parameters of the auxiliary pdf family as

$$\lambda_k = \underset{\lambda\in\Delta}{\operatorname{argmax}} \left\{ \frac{1}{N}\sum_{i=1}^{N} \left[\mathbf{1}_{\phi(\mathbf{X}_i^{(k)})>q_k} \frac{h_0(\mathbf{X}_i^{(k)})}{h_{\lambda_{k-1}}(\mathbf{X}_i^{(k)})} \ln\left[h_\lambda(\mathbf{X}_i^{(k)})\right] \right] \right\}$$

(v) If $q_k < S$, $k \leftarrow k+1$, go to step (ii), else
(vi) Estimate the probability

$$\widehat{P}^{CE}(\phi(\mathbf{X}) > S) = \frac{1}{N}\sum_{i=1}^{N} \mathbf{1}_{\phi(\mathbf{X}_i^{(k)})>S} \frac{h_0(\mathbf{X}_i^{(k)})}{h_{\lambda_{k-1}}(\mathbf{X}_i^{(k)})}$$

2.2.3. Non-parametric adaptive importance sampling

The objective of non-parametric adaptive importance sampling (NAIS) technique is to approximate the IS optimal auxiliary density given in equation (2) with kernel based probability distributions. The complete details of this algorithm can be found in [2,16]. Using Gaussian kernel densities, the NAIS algorithm is given by

(i) $k = 1$ and set $\rho \in (0, 1)$
(ii) Generate the population $\mathbf{X}_1^{(k)}, \ldots, \mathbf{X}_N^{(k)}$ according to the pdf $h_{k-1}(\cdot)$, apply the function $\phi(\cdot)$ in order to have $Y_1^{(k)} = \phi(\mathbf{X}_1^{(k)}), \ldots, Y_N^{(k)} = \phi(\mathbf{X}_N^{(k)})$
(iii) Compute the empirical ρ-quantile $q_k = Y^{(k)}_{\lfloor \rho N \rfloor}$
(iv) Estimate $I_k = \frac{1}{kN}\sum_{j=1}^{k}\sum_{i=1}^{N} \mathbf{1}_{\phi(\mathbf{X}_i^{(j)})\geq q_k} \frac{h_0(\mathbf{X}_i^{(j)})}{h_{j-1}(\mathbf{X}_i^{(j)})}$
(v) Update the Gaussian kernel sampling pdf with

$$h_k(\mathbf{X}) = \frac{1}{kNI_k \det(B_k)} \sum_{j=1}^{k}\sum_{i=1}^{N} w_j(\mathbf{X}_i^{(j)}) K_d\left(B_k^{-1}\left(\mathbf{X} - \mathbf{X}_i^{(j)}\right)\right),$$

where K_d is the standard d-dimensional Gaussian function with zero mean and a diagonal covariance matrix $B_k = diag(b_k^1, \ldots, b_k^d)$ and $w_j\left(\mathbf{X}_i^{(j)}\right) = \mathbf{1}_{\phi(\mathbf{X}_i^{(j)})\geq q_k} \frac{h_0(\mathbf{X}_i^{(j)})}{h_{j-1}(\mathbf{X}_i^{(j)})}$. The coefficients of matrix B_k are optimized according to the AMISE (Asymptotic Mean Integrated Square Error) criterion [17,18].
(vi) If $q_k < S$, $k \leftarrow k+1$, go to step (ii), else
(vii) Estimate the probability

$$\widehat{P}^{NAIS}(\phi(\mathbf{X}) > S) = \frac{1}{N}\sum_{i=1}^{N} 1_{\phi(\mathbf{X}_i^{(k)})>S} \frac{h_0(\mathbf{X}_i^{(k)})}{h_{k-1}(\mathbf{X}_i^{(k)})}$$

CE and NAIS simulation techniques are generally efficient to estimate accurately a rare event probability in a large range of realistic situations [16].