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Optimal representation of multi-dimensional random fields with a moderate number of samples: Application to stochastic mechanics

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ABSTRACT

A significant amount of problems and applications in stochastic mechanics and engineering involve multi-dimensional random functions. The probabilistic analysis of these problems is usually computationally very expensive if a brute-force Monte Carlo method is used. Thus, a technique for the optimal selection of a moderate number of samples effectively representing the entire space of sample realizations is of paramount importance. Functional Quantization is a novel technique that has been proven to provide optimal approximations of random functions using a predetermined number of representative samples. The methodology is very easy to implement and it has been shown to work effectively for stationary and non-stationary one-dimensional random functions. This paper discusses the application of the Functional Quantization approach to the domain of multi-dimensional random functions and the applicability is demonstrated for the case of a 2D non-Gaussian field and a two-dimensional panel with uncertain Young modulus under plane stress.

1. Introduction

In stochastic engineering problems, the proper consideration of the input variability is crucial to obtain an accurate and reliable solution. A large number of these problems involves uncertain quantities which should be modeled as multi-dimensional random fields. The use of multi-dimensional random fields gained momentum due to the continued increase in available computational resources and nowadays is commonly used in many disciplines. Several examples can be found in various fields of engineering. For instance, in structural engineering Christou and Bocchini [1] modeled the spatial distribution of corrosion over the upper flange of a steel I-beam as a two dimensional random field. Papadopoulos and Papadrakakis [2] used two-dimensional uni-variate (2D-1V) stochastic fields to describe the non-homogeneous characteristics of initial imperfections in manufactured shells. In geotechnical engineering Popescu et al. [3] used two-dimensional fields to model the spatial variability of the soil mechanical characteristics. Similarly, in naval engineering Teixeira and Guedes Soares [4] used two-dimensional fields to model the spatial corrosion propagation in ship-hull plates and computed their collapse strength.

The solution of these engineering problems is often obtained though simulation-based techniques, which are the most commonly used among the procedures available in the literature. Monte Carlo Simulation (MCS) is still considered the most reliable and versatile numerical technique for the solution of engineering problems affected by uncertainty. The drawback of MCS remains the large computational cost that prevents its use for many applications. In particular, the number N of deterministic runs which can be actually performed is limited by the complexity of the problem at hand, the type of input and the available time and computational resources. In many cases, this number N is small, e.g. in the range [50–1000], too small for the law of large numbers to apply. Such sample size may be sufficient for the assessment of low-order statistics (i.e., mean or standard deviation at most), but certainly not to capture more information on the probability distribution. Thus, the result of a plain MCS would not be acceptable when a model of the entire distribution is sought. In these cases, a probabilistic technique that can capture in the most effective way the space of sample realizations of the random function, given a pre-determined number of samples, should be used.

Multiple techniques have addressed the issue of sampling random functions more effectively, compared to plain MCS. For engineering problems the currently most popular method was presented in [5] and is called "Stochastic Reduced-Order Models" (SROM). Grigoriu used SROM to find statistics of the state of linear dynamic systems with random and deterministic properties

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Nomenclature

Notation	Definition
1_{Ω_i}	indicator function of Ω_i
Δ	distortion
$\Delta\kappa_1$, $\Delta\kappa_2$	space increments
$\Delta\xi_1$, $\Delta\xi_2$	wavenumber increment
Σ	covariance matrix
Ξ	spatial domain of interest in $\mathbb{R}^n$
Ω	sample space
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
κ_1, κ_2	wave number in the two investigated dimensions
λ	correlation length
$\mu_{R_{vv}}$	mean value of $R_{vv}(v_{111}, v_{61})$ for all the estimations
$\bar{\mu}_{R_{vv}}$	mean value of $R_{vv}(v_{111}, v_{61})$ computed from MCS
ξ_1, ξ_2	field axes in the spatial domain
$\rho(\mathbf{y})$	mass density at point $\mathbf{y}$
BE	bias error
CVT	Centroidal Voronoi Tesselation
D_{KS}	Kolmogorov–Smirnov index
F	random function
F_N	random function used to approximate F
FQ	Functional Quantization
$F(\boldsymbol{\xi}, \omega)$	bi-measurable random field
IDCVT	Infinite-Dimensional Centroidal Voronoi Tesselation
$L^2(\Xi)$	space of square integrable functions
L_1, L_2	number of discretization points in the wavenumber domain
M_1, M_2	number of points of the simulated stochastic field
MCS	Monte Carlo simulation
N	quantizer size
nseeds	number of different seeds used to obtain different estimations
N_i	number of samples f_i that belong to tassel V_i
N_{sim}	computational parameter ($N_{sim} = N \cdot k$, where $k \in [100, 5000]$)
$\mathbb{P}(\Omega_i)$	probability associated with the subset Ω_i
$\mathbb{P}_F(V_i)$	probability mass of tassel V_i
PDF(x)	probability density function
R_{vv}	autocorrelation between two points
$\mathbb{R}^n$	Euclidean n-space
SDF	spectral density function
SE	stochastic error
S_{FF}	spectral density function
SRM	spectral representation method
T_{ξ_1}, T_{ξ_2}	periods of the generated field
$\{V_i\}_{i=1}^N$	tassels corresponding to $\{\Omega\}_{i=1}^N$
VT	Voronoi Tesselation
b_1, b_2	parameters proportional to the correlation length
f_i	deterministic function representative of F_N over Ω_i (i.e., quanta)
$\check{f}_i$	generating point of tassel V_i
$\bar{f}_i$	centroid of tassel V_i
$\{f_i, p_i\}_{i=1}^N$	quantizer
$\{\hat{f}_k\}_{k=1}^{N_i}$	represents the N_i samples that belong to tassel V_i
k_{u_1}, k_{u_2}	upper cutoff frequency
p_i	probability mass
$\boldsymbol{\xi}$	point in Ξ
y_i	all points that belong to V_i (finite-dimensional case)
$\check{y}$	generating point of tassel V_i (finite-dimensional case)
$\bar{y}$	mass centroid of the tassel (finite-dimensional case)

subjected to Gaussian and non-Gaussian noise [6,7]. Mingolet and Soize [8] utilized SROM for the determination of the response of geometrically nonlinear structural dynamic systems and Warner et al. [9] employed it to approximate the natural frequencies and modes of uncertain dynamic systems. The basic idea of SROM methods is to consider an optimization problem where the objective function quantifies the discrepancy between the statistics of the SROM and the random function being modeled. It will be shown that the methodology presented in this paper is rooted in a similar idea, but the optimization problem will be formulated in a different way. A discussion of the differences between SROM and the proposed methodology is provided in Section 4.

As an alternative to SROM, Functional Quantization (FQ) is a novel technique, proven to provide optimal approximations of random functions using a pre-determined number N of representative samples [10,11]. Moreover, some authors have used FQ directly as a variance reduction technique [12,13]. FQ is characterized by two major differences, compared to MCS: (1) the representative samples from FQ are selected not entirely at random and (2) the representative samples from FQ are not equally weighted. A few techniques for the selection of optimal samples and computation of associated probabilities based on the FQ concept have been presented in the literature. Some of the best known quantization techniques were presented by Lushgy and Pagés under the "Quantizer Design" umbrella [14]. The Quantizer Design I yields optimal results, whereas Quantizer Design II, III and IV are sub-optimal, but they are characterized by improved computational efficiency. All these techniques rely on the use of Karhunen–Loéve expansions, and therefore, they have been demonstrated only on Gaussian random functions for which such expansion is readily available. Another class of techniques was proposed by Corlay and Pagés [15]. They have the appeal of connecting FQ with the very popular stratified sampling approach. The authors presented four different versions of the approach but as in the case of the Quantizer Design, these techniques have been applied only to one-dimensional Gaussian processes.

To overcome the limitations that affect the previously mentioned FQ techniques, in this paper, a recently developed methodology called "Functional Quantization by Infinite-Dimensional Centroidal Voronoi Tessellation" (FQ-IDCVT) is considered [16]. The FQ-IDCVT technique has been successfully used for one-dimensional, non-Gaussian and non-stationary processes [17] and it has been shown to work particularly well against the curse of dimensionality that arises in stochastic problems that use random functions for input quantities.

To further enhance the versatility of the technique, this paper presents it's extension to the case of random functions defined over a multi-dimensional domain. A description of the modifications to the existing algorithm required by the extension to multi-dimensional fields is accompanied by its demonstration on two numerical applications. The first example involves a two-dimensional lognormal field, generated through the Spectral Representation method (SRM) [18]. For the simulation of the two dimensional field, a recently developed algorithm that approximates a non-Gaussian stationary multi-dimensional field is utilized [1]. Next, a sensitivity analysis is used to discuss the limits of applicability of the proposed approach and comments on the computational challenges associated with the extension to two-dimensional fields are provided. Finally, a second numerical application involving a two-dimensional panel in plane-stress with