Probabilistic Engineering Mechanics 44 (2016) 111–117

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Probability density evolution method: Background, significance and recent developments

Jie Li

State Key Laboratory of Disaster Reduction in Civil Engineering & School of Civil Engineering, Tongji University, Shanghai 200092, PR China

ARTICLE INFO

Article history:
Received 20 August 2015
Accepted 29 September 2015
Available online 21 October 2015

Keywords:
Probability density evolution method
Generalized probability density evolution equation
Probability dissipation
Stochastic system
Dynamic reliability
Structural stability

ABSTRACT

Investigation of a stochastic system from a viewpoint of studying the randomness propagation process in a physical system starts to play an important role in understanding the complete performance or behaviour of engineering systems. Following this path, the probability density evolution method (PDEM) has been developed by Li and Chen at the beginning of this century on the basis of the principle of probability preservation and its random event description. A family of generalized probability density evolution equation (GPDEE) was then derived. The systematic analysis indicates that the new family of equation actually reveals the logical fundamentals of randomness propagation in a physical system: the transition of the probabilistic structure in a stochastic system definitely relies on the change of physical state of the system. This paper devotes to a summary on the background and basic theoretical developments of the PDEM. Considering the limitation of probability conservative systems for the GPDEE, a novel concept of probability dissipation is introduced and a completely uncoupled partial differential equation is derived as well with respect to the evolutionary probability density function, which holds for any physical quantity of a probability dissipative system. For illustrative purposes, several engineering applications, including the dynamic reliability assessment of controlled structures with fractional derivative viscoelastic dampers and the stability analysis of structures under dynamic loading, are investigated, respectively.

1. Introduction

The great achievements of modern science and technology in the 20th century have strongly enhanced the ability of mankind to understand the phenomena, performance and mechanism of engineering systems. As time goes on, the deficiency and inadequacy of deterministic methodologies in modelling and analysis of engineering systems are recognized more and more clearly, and therefore, searching for a physical approach to stochastic systems started to play an important role in understanding uncertain behaviours and performance of engineering systems, particularly those emerging in nonlinear systems. In this way, the idea of randomness propagation in a physical system starts to gain increasing attention from researchers with interest focused on stochastic dynamical systems.

In history, the study on stochastic dynamical systems was generally recognized to be originated from Einstein's investigation on Brownian motion. In 1905, by introducing the irregular collisions between the Brownian particles, Einstein deduced an evolution equation of the density of particles and found that this equation was a diffusion equation [5]. This thought was then boosted by Fokker in 1914 and by further Planck in 1917, leading to the probability density equation well known as the Fokker–Planck equation in the physicist community [32,6]. In 1931, the Soviet mathematician Kolmogorov derived the same equation independently. This investigation set a rigorous mathematical foundation for the equation [10]. It is noticeable that although Einstein started with the physical mechanism of irregular collisions of molecules on Brownian particles, the crux of his tactics is to treat the bulk behaviour of particles evolution in a phenomenological way. Due to Kolmogorov's work, the analysis of stochastic dynamical systems can be converted to the problem of a deterministic partial differential equation. Afterwards, far more emphasis was put on the mathematical aspects than on the physical aspects.

Shortly later than Einstein, to study Brownian motion, another famous physicist Langevin applied Newton's law to a single Brownian particle and obtained the dissipation–diffusion relation identical to Einstein's result in a much more concise way [12]. In Langevin's investigation, the resultant force induced by the collisions of the around molecules in the fluids was regarded as an irregular (random) force exerted on the Brownian particle. This treatment was so impressing that scientists believed that

E-mail address: lijie@tongji.edu.cn

http://dx.doi.org/10.1016/j.probengmech.2015.09.013
0266-8920/

Langevin's method is an effective and independent new method though the assumptions on the irregular force were somewhat baffling. In the early 1920s, Wiener studied the features of Brownian motion and built the foundation for correctly understanding the meanings of Langevin's assumptions [36]. In the early and middle 1940s, Itô introduced rigorous definition of the Itô calculus and therefore clarified that the Langevin's forces could be modelled by the mathematical white noise [8,9]. In the early 1960s, Stratonovich came up with the physical interpretation of the white noise [35].

The methodology originated from Einstein, along the path of Einstein–Fokker–Planck–Kolmogorov might be referred to as the phenomenological tradition in the studies of stochastic dynamical systems. On the other hand, in the methodology originated from Langevin, along the path of Langevin–Wiener–Itô–Stratonovich, may be referred to as the physical tradition in stochastic dynamics [14,15].

The cross of the phenomenological tradition and the physical tradition consists a reasonable starting point for modern studies on stochastic systems. In the early 1950s, the random vibration theory, a branch of stochastic dynamics, was developed gradually as an independent branch of engineering science [4], and then was applied to different engineering disciplines such as civil, mechanical, aeronautics and aerospace, and marine and offshore engineering, etc. [27,28,30]. Till the early 1990s, the theory and pragmatic approaches for random vibration of linear structures were well developed. However, for nonlinear systems, despite great efforts devoted since 1960s, including a variety of methods such as the stochastic linearization method, equivalent non-linearization method, stochastic averaging method, cumulant-neglect closure method, path-integration method, Hamiltonian formulation and so on [11,27,28,38,39], there are still great challenges for the non-stationary response analysis of general nonlinear dynamical systems, particularly for MDOF systems [31,33,7]. Under such background, in early this century, Li and Chen found that the principle of probability preservation provides a unified logic foundation for investigating the randomness propagation in stochastic dynamical systems. Based on the clarification of the principle of probability preservation, especially from the state description and random event description, a new family of general probability density evolution equation (GPDEE) was developed [16–18], which could secure the randomness propagation in a dynamical system. Actually, this investigation reveals that it is just the state evolution of a physical system induced by the physical laws resulting in the probability density evolution of the corresponding stochastic system [19]. This understanding or new finding therefore established a straightforward relationship between general physical systems and the corresponding stochastic systems.

Owing to the above background, this paper contributes to a summary upon the basic theoretical foundation of the probability density evolution method and recent developments including its extension to general physical systems and even probability dissipative systems. Several engineering applications are investigated for illustrative purposes, including the dynamic reliability assessment of controlled structures with fractional derivative viscoelastic dampers and the stability analysis of structures under dynamic loading, respectively.

2. Theoretical foundation of probability density evolution method

The principle of probability preservation can be stated as follows: if the random factors involved in a stochastic system are retained, then the probability will be preserved in the state evolution process of the system [20]. In order to clarify the principle, let us start with the investigation on a transform of a random function.

Let ϖ be a basic random event and $X(\varpi)$ be a continuous variable with probability density function (PDF) $p_X(x)$, namely

$$\Pr\{X(\varpi) \in (x, x+dx)\} = d\Pr\{\varpi\} = p_X(x)dx \tag{1}$$

where $\Pr\{\cdot\}$ is the probability measure.

Assume there exists a one to one mapping from X to Y, that is

$$f: Y = f(X) \tag{2}$$

then the PDF of Y will be

$$p_Y(y) = p_X[f^{-1}(y)]\frac{dx}{dy} \tag{3}$$

Obviously, the above equation could be converted to

$$p_Y(y)dy = p_X(x)dx \tag{4}$$

Noticing that

$$\Pr\{Y(\varpi) \in (y, y+dy)\} = d\Pr\{\varpi\} = p_Y(y)dy \tag{5}$$

it is evident that

$$\Pr\{Y(\varpi) \in (y, y+dy)\} = \Pr\{X(\varpi) \in (x, x+dx)\} = d\Pr\{\varpi\} \tag{6}$$

This means that, in a mathematical transform, the probability measure will be preserved since the random events keep unchanged. This statement reveals the principle of probability preservation. The principle is universally applicable to generic stochastic systems.

For a general multi-degree-of-freedom (MDOF) system, the basic equation of motion will be

$$\mathbf{M}(\boldsymbol{\eta})\ddot{\mathbf{X}} + \mathbf{C}(\boldsymbol{\eta})\dot{\mathbf{X}} + \mathbf{f}(\boldsymbol{\eta}, \mathbf{X}) = \boldsymbol{\Gamma}\xi(t) \tag{7}$$

where $\boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_{s_1})$ are the random parameters involved in the physical properties of the system with known joint PDF; $\ddot{\mathbf{X}}, \dot{\mathbf{X}}, \mathbf{X}$ are the accelerations, velocities and displacements of the system, respectively; $\mathbf{M}, \mathbf{C}$ are the mass and damping matrices of the system; $\mathbf{f}(\cdot)$ is the nonlinear restoring force vector; $\boldsymbol{\Gamma}$ is an n_d-dimensional column vector and $\xi(t)$ is a stochastic excitation.

For general stochastic processes, some mathematical representations, e.g., the Karhunen–Loève decomposition [29] or the spectral representation method [34], can be adopted such that the excitation could be represented by a random function

$$\xi(t) = \xi(\boldsymbol{\zeta}, t) \tag{8}$$

where $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_{s2})$ is a set of independent random variables with known joint PDF.

For notational clarity, denoting

$$\boldsymbol{\Theta} = (\boldsymbol{\eta}, \boldsymbol{\zeta}) = (\eta_1, \eta_2, \ldots, \eta_{s_1}, \zeta_1, \zeta_2, \ldots, \zeta_{s2}) = (\Theta_1, \Theta_2, \ldots, \Theta_s) \tag{9}$$

in which $s = s_1 + s_2$ is the total number of the basic random variables involved in the system, then Eq. (7) can be rewritten into

$$\mathbf{M}(\boldsymbol{\Theta})\ddot{\mathbf{X}} + \mathbf{C}(\boldsymbol{\Theta})\dot{\mathbf{X}} + \mathbf{f}(\boldsymbol{\Theta}, \mathbf{X}) = \mathbf{F}(\boldsymbol{\Theta}, t) \tag{10}$$

where $\mathbf{F}(\boldsymbol{\Theta}, t) = \boldsymbol{\Gamma}\xi(\boldsymbol{\zeta}, t)$.

Eq. (9) is a general stochastic differential equation in which all the randomness from the initial conditions, excitations and system parameters is involved and exposed in a unified manner.

If, besides the displacement and velocity, other physical quantities $\mathbf{Z}(t) = (Z_1, Z_2, \ldots, Z_m)^{\mathrm{T}}$ of the system are also interested, the following relationship holds:

$$\mathbf{Z}(t) = \boldsymbol{\Psi}[\mathbf{X}(t), \dot{\mathbf{X}}(t)] \tag{11}$$

where $\boldsymbol{\Psi}$ denotes the operator that converts the state vector to the