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An integral transform approach for solving partial differential equations with stochastic excitation

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ABSTRACT

In this paper an integral transform approach for solving a class of initial-boundary-value problems involving linear stochastic partial differential equations (SPDEs) encountered in engineering mechanics applications is presented. The stochastic solution is represented in the complex z-plane, and is inspired by the corresponding approach to deterministic PDEs already established in the literatures. In this regard, it is noted that the determination of relevant correlation function and spectral density of the solution is expedited because of the induced separability of the displacement x, and the time t in their representations. For three kinds of SPDEs on the half-line $\{0 < x < \infty, t > 0\}$ involving a spatially dependent white noise excitation, relevant response statistics are determined. The influence of the boundary condition on the correlation function of the response is also discussed. Further, the corresponding spectral density is also expressed as an integral in the complex z-plane using the Wiener–Khintchine relation. Numerical examples pertaining to the stochastic heat equation suggest that the new transform approach is a viable tool of analysis.

1. Introduction

Applications of stochastic partial differential equations (SPDEs) in the field of mechanics are well established in [1–6], etc. A rather exhaustive bibliography on SPDEs may be found in [7].

Clearly, determining the general solutions for SPDEs is a challenging task, especially in the presence of nonlinearity or for complex boundary conditions [1,2,6]. Note that the basic correlation function and spectral density calculations for stochastic response become even more difficult, in view of the inseparability of the displacement x and the time t in the representation of the stochastic response.

In this regard, a novel transform method, the so-called "unified transform method", described by [8–10] has been found to be effective for solving certain deterministic linear PDEs. The method has also been found to be conveniently applicable to multi-space variables and high-order PDEs.

The method can be understood as "*synthesis, as opposed to separation, of variables*" (in contrast to the classical separation of variables which is limited in boundary conditions for solving certain PDEs [11]. Therein, the method can also be applied even to certain non-separable and non-self-adjoint problems.

Specifically, the solutions of certain linear PDEs can be represented by an integral in the complex z-plane on the half line $\{0 < x < \infty, t > 0\}$ [12]. The aforementioned work is fundamental background for the treatment of SPDEs pursued in this paper. In this regard, the integral representation of the stochastic response with separability of the displacement x and the time t defined by the transform approach will be helpful to determine the correlation function and spectral density of the stochastic response.

2. Stochastic linear evolution PDEs

2.1. Spatially dependent white noise

In the ensuing analysis, two kinds of spatially dependent white noises are considered as the stochastic excitations. The first one is

$$\xi(x, t) = r(x)\eta(t), \tag{1}$$

in which $r(x)$ is sufficiently smooth and deterministic function. The symbol $\eta(t)$ denotes a Gaussian white noise with

$$\langle\eta(t)\rangle = 0, \ \langle\eta(t)\eta(s)\rangle = D\delta(t - s), \tag{2}$$

where D is the intensity of $\eta(t)$, $\langle\bullet\rangle$ denotes statistical averaging, and $\delta(\bullet)$ is Dirac Delta function.

The second spatially dependent white noise is

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$$\xi(x,t) \doteq \dot{W}(x,t), \tag{3}$$

which can be seen as the formal time derivative $\frac{\partial}{\partial t}W(x,t)$ of the Wiener random field $W(x,t)$ with conditions [2]

$$\left\langle \dot{W}(x,t) \right\rangle = 0, \tag{4}$$

and

$$\langle \dot{W}(x,t)\dot{W}(y,s)\rangle = D\delta(t-s)\delta(x-y). \tag{5}$$

2.2. General nonhomogeneous SPDE

Consider a general linear evolution equation on the half-line with stochastic excitation $\xi(x,t)$ introduced in Section 2.1. That is,

$$\partial_t u(x,t) + \varnothing(-i\partial_x)u(x,t) = \xi(x,t), \tag{6}$$

with

$$u(x,0) = u_0(x),\ \partial_x^j u(0,t) = g_j(t), \tag{7}$$

and

$$0<x<\infty, t>0, j=0,1,\cdots,N-1, \tag{8}$$

where the symbol $\varnothing(z)$ is an arbitrary polynomial; the function $u_0(x)$ is sufficiently smooth; $g_j(t)$ is a function of the time t including certain stochastic process; and N is an integer number.

For instance, three popular equations can be retrieved from Eq. (6) given $\varnothing(z)$ and $N = 1$. Specifically, the stochastic heat equation

$$u_t = u_{xx} + \xi(x,t),\ for\ \varnothing(z) = z^2; \tag{9}$$

the stochastic Stokes equation

$$u_t + u_{xxx} = \xi(x,t),\ for \varnothing(z) = -iz^3; \tag{10}$$

and the Schrödinger equation with zero potential

$$iu_t + u_{xx} = \xi(x,t),\ for\ \varnothing(z) = iz^2. \tag{11}$$

2.3. The representation of solution for certain linear PDE by the transform approach

The integral transform approach can be applied to any linear deterministic homogeneous PDEs with constant coefficients and certain initial and boundary conditions [8,10,12]. For instance, consider a linear evolution homogeneous PDE on the half-line. That is,

$$\partial_t q(x,t) + \varnothing(-i\partial_x)q(x,t) = 0, \tag{12}$$

and

$$q(x,0) = q_0(x),\ \partial_x^j q(0,t) = \tilde{g}_j(t), \tag{13}$$

with

$$0<x<\infty, t>0, j=0,1,\cdots,N-1. \tag{14}$$

Here the solution $q(x,t)$ can be expressed via one parameter family satisfying the equation for any complex z,

$$\left(e^{-izx+\varnothing(z)t}q\right)_t + \left(e^{-izx+\varnothing(z)t}X\right)_x = 0, \tag{15}$$

where

$$X(x,t,z) = -i\frac{\varnothing(z)-\varnothing(l)}{z-l}q, \tag{16}$$

with $l = -i\partial_x$. Specifically, for Eq. (9),

$$X(x,t,z) = -izq - q_x; \tag{17}$$

for Eq. (10),

$$X(x,t,z) = -z^2q + izq_x + q_{xx}; \tag{18}$$

and for Eq. (11),

$$X(x,t,z) = zq - iq_x. \tag{19}$$

Next, the unknown information in the boundary conditions for constructing the solution $q(x,t)$ can be determined from the relation

$$\hat{g}(z,t) = -\hat{q}_0(z), \mathrm{Im}\, z<0, \tag{20}$$

where $\hat{q}_0(z)$ and $\hat{g}(z,t)$ are represented by

$$\hat{q}_0(z) = \int_0^\infty dx e^{-izx} q(x,0), \tag{21}$$

and

$$\hat{g}(z,t) = \int_0^t d\tau e^{\phi(z)\tau} X(0,\tau,z). \tag{22}$$

At the boundary $x = 0$, the definition of $\hat{g}_j(z,t)$ is denoted by

$$\hat{g}_j(z,t) = \int_0^t d\tau e^{\varnothing(z)\tau}\tilde{g}_j(\tau), \tag{23}$$

$j = 0, 1, \cdots, N-1$.

Therefore, using Eqs. (20)–(23), the function $\hat{g}(z,t)$ in Eq. (22) can be determined by eliminating the unknown term $\hat{g}_j(z,t)$. For convenience, if $N = 1$, the final expression of $\hat{g}(z,t)$ can be separated into

$$\hat{g}(z,t) = f_\varnothing(z)\hat{g}_0(z,t) + \hat{\rho}(z), \tag{24}$$

where

$$f_\varnothing(z) = -2iz,\ \mathrm{for} \varnothing(z) = z^2, \tag{25}$$

$$f_\varnothing(z) = -3z^2,\ for \varnothing(z) = -iz^3, \tag{26}$$

and

$$f_\varnothing(z) = 2z,\ for \varnothing(z) = iz^2. \tag{27}$$

The function $\hat{\rho}(z)$ is the remaining part of $\hat{g}(z,t)$ which is independent of the time t.

The solution of Eq. (12) can then be expressed in the form [8,12]

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz e^{izx-\varnothing(z)t}\hat{q}_0(z) + \frac{1}{2\pi}\int_{-\infty}^{\infty} dz e^{izx-\varnothing(z)t}\hat{g}(z,t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dz e^{izx-\varnothing(z)t}\hat{q}_0(z) + \frac{1}{2\pi}\int_{\partial D_+} dz e^{izx-\varnothing(z)t}\hat{g}(z,t). \tag{28}$$

where $\hat{g}(z,t)$ is given by Eq. (24). Here introduce $D_+ = \{z \in C\text{: Re}\,\varnothing(z)<0, \mathrm{Im}\, z>0\}$; and ∂D_+ the boundary of this domain oriented so that D_+ is on the left of ∂D_+. Note that the integrands in Eq. (28) are analytical and bounded in the domain of $E_+ = \{z \in C\text{: Re}\,\varnothing(z)>0, \mathrm{Im}\, z>0\}$.

2.4. General solution of Eqs. (6)–(8) with stochastic excitation (1) or (3)

The stochastic excitation $\xi(x,t)$ in Eq. (1) or (3) is assumed to be sufficiently smooth in x-axis, stationary and mean square integrable with zero mean value. That is, $\xi(x,t)$ satisfies the condition