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# A nonlinear weakly dispersive method for recovering the elevation of irrotational surface waves from pressure measurements

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A B S T R A C T

We present the derivation of a nonlinear weakly dispersive formula to reconstruct, from pressure measurements, the surface elevation of nonlinear waves propagating in shallow water. The formula is simple and easy to use as it is local in time and only involves first and second order time derivatives of the measured pressure. This novel approach is evaluated on laboratory and field data of shoaling waves near the breaking point. Unlike linear methods, the nonlinear formula is able to reproduce at the individual wave scale the peaked and skewed shape of nonlinear waves close to the breaking point. Improvements in the frequency domain are also observed as the new method is able to accurately predict surface wave elevation spectra over four harmonics. The nonlinear weakly dispersive formula derived in this paper represents an economic and easy to use alternative to direct wave elevation measurement methods (e.g. acoustic surface tracking and LiDAR scanning).

## 1. Introduction

Near-bottom-mounted pressure sensors have long been used for measuring surface wave in the nearshore. However, the relationship between bottom pressure and sea surface elevation is not straightforward. This relationship is commonly assumed to be given by linear wave theory, the so-called transfer function method (e.g. Bishop and Donelan (1987) and Tsai et al. (2005)). The validity of this linear reconstruction has been extensively studied in field conditions for waves propagating in relatively shallow water (Hom-ma et al., 1966; Esteva and Harris, 1970; Cavaleri et al., 1978; Guza and Thornton, 1980). Although discrepancies were greater close to the break point, Guza and Thornton (1980) found a good agreement in and outside the surf zone between sea surface elevation spectra derived from pressure data and from direct elevation measurements. Errors in both total variance and energy density in a particular frequency band were less than 20%. In a more controlled environment, Bishop and Donelan (1987) estimated that using linear wave theory was leading to error of about 5% of the wave height; uncertainty in the deployment of *in situ* instruments and the data itself was thought to be responsible for the varying error estimates found in the literature. Following these seminal studies, the linear reconstruction method has become the main approach for characterizing shallow-water surface-wave elevation in field conditions.

This approach is commonly used for determining bulk wave parameters such as the significant wave height $H_s$, but it has also served as a basis for studying nonlinear wave interactions in the field (e.g. Elgar and Guza (1985), Elgar et al. (1997), Senechal et al. (2002), Henderson et al. (2006)). However, we know that wave nonlinearities can be strong in the shoaling zone, especially in the region close to the onset of breaking, and thus the use of a linear theory to reconstruct wave elevation can be questioned. For instance, Bonneton and Lannes (2017) and Martins et al. (2017a) have shown that the linear reconstruction fails to describe the peaky and skewed shape of nonlinear waves, and lead to an underestimation of the individual wave height by up to 30% just prior the breaking point (Martins et al., 2017a). Such measurement errors are problematic for many coastal applications, such as studies on wave overtopping and submersion which require accurate measurements of the highest wave crests. Furthermore, a correct description of wave asymmetry and skewness is of paramount importance for understanding sediment dynamics (e.g. Dubarbier et al. (2015)). Finally, an accurate description of the wave elevation field is also crucial for the validation of the new generation of fully-nonlinear phase-resolving wave models (e.g. Zijlema et al. (2011), Bonneton et al. (2011) or Shi et al. (2012)).

Even if some methods are now available for a direct measurement of

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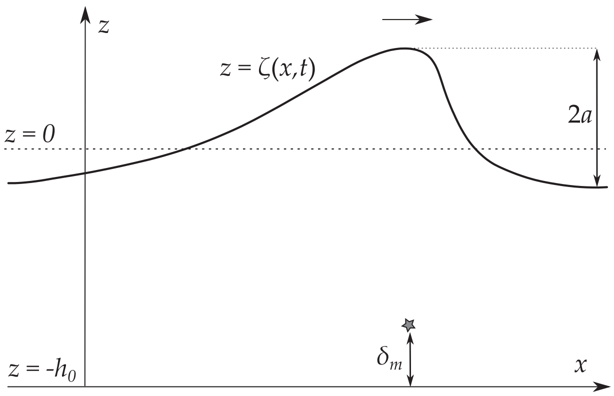


**Fig. 1.** Sketch of the Cartesian coordinate system: $x$ is the horizontal axis along which waves propagate and $z$ points vertically upwards, with $z=0$ being the mean water level and $z=-h_0$ the distance to the bottom. The wave amplitude is noted $a$ and $\delta_m$ represents the distance to the bottom at which the pressure is measured.

the surface elevation, such as acoustic surface tracking (Birch et al., 2004) or LiDAR scanning (Martins et al., 2016), pressures sensors remain a very useful tool for coastal wave applications. Indeed, they are cheap, robust, not sensitive to air bubbles or turbidity, and easy to deploy since they do not require the presence of nearshore infrastructure, as it can be the case for LiDAR technology [e.g., see (Martins et al., 2017c)]. Bonneton and Lannes (2017) recently derived a method which allows a fully dispersive nonlinear reconstruction of the surface elevation from pressure measurements. Comparisons with numerical Euler solutions and laboratory data showed that this nonlinear method provides much better results than the classical linear approach. It gives an accurate prediction of the maximum elevation and, contrary to the nonlinear heuristic method proposed by Oliveras et al. (2012), it accurately reproduces the skewed shape of nonlinear dispersive wave fields. However, this method requires, like the classical linear transfer approach (Bishop and Donelan, 1987; Tsai et al., 2005) and the heuristic method (Oliveras et al., 2012; Vasan et al., 2017), the use of a frequency cutoff which becomes a limiting factor for the reconstruction of strongly nonlinear waves. In the present paper we derive a nonlinear weakly-dispersive method which allows an accurate reconstruction of nonlinear waves in shallow water, especially just prior to breaking.

## 2. Nonlinear weakly-dispersive reconstruction formula

In this section we derive a formula working in the time domain, which allows the elevation reconstruction of nonlinear shallow water waves from pressure measurements. This formula is an approximate expression, in the shallow water regime, of the fully dispersive formula derived in (Bonneton and Lannes, 2017). The derivation presented in this section is much simpler and straightforward compared to the general fully dispersive derivation.

We consider that the wave field is locally close to a two-dimensional wave field. We choose Cartesian coordinates $(x,z)$, where $x$ is the horizontal axis along which waves propagate and $z$ the upward vertical coordinate. We denote $z=\zeta(x,t)$ the elevation of the free surface above the still water level $z=0$, and by $z=-h_0$ the constant bottom elevation (see Fig. 1). The water depth $h$ can be expressed as $h(x,t)=h_0+\zeta(x,t)$. The pressure $P_{\mathrm{m}}$ is measured at a distance $\delta_m$ above the bottom, $P_{\mathrm{m}}=P_{|_{z=-h_0+\delta_m}}$, where $P(x,z,t)$ is the pressure field.

The fluid motion is governed by the free-surface incompressible irrotational Euler equations; if the flow is irrotational, as it is assumed here, it is convenient to work with a velocity potential instead of the velocity field, and with Bernoulli's equation instead of Euler equations. If $\phi$ denotes the velocity potential, these equations can be recast in the form:

$$\partial_x^2\phi+\partial_z^2\phi=0$$

$$\partial_t\phi+gz+\frac{1}{2}|\partial_x\phi|^2+\frac{1}{2}|\partial_z\phi|^2=-\frac{1}{\rho}(P-P_{atm}),$$

where $\rho$ is the water density, $g$ the gravity and $P_{atm}$ the (constant) atmospheric pressure. These equations are complemented by boundary conditions. At the bottom we have

$$\partial_z\phi=0 \quad \text{on } z=-h_0; \tag{1}$$

at the surface, we have the classical kinematic equation on $\zeta$,

$$\partial_t\zeta=\partial_z\phi-\partial_x\zeta\cdot\partial_x\phi \quad \text{on} \quad z=\zeta, \tag{2}$$

and the pressure continuity,

$$P=P_{\mathrm{atm}} \quad \text{on} \quad z=\zeta. \tag{3}$$

Three main length scales are involved in this problem: the characteristic horizontal length $L$ ($L=1/k$, where $k$ is the typical wave number), the amplitude $a$ of the wave, and the depth at rest $h_0$. The problem is then controlled by two dimensionless parameters:

$$\varepsilon=\frac{a}{h_0}, \qquad \mu=\frac{h_0^2}{L^2}=(h_0k)^2$$

where $\varepsilon$ is a nonlinearity parameter while $\mu$ is the shallowness parameter. The different variables and functions involved in this problem can be put in dimensionless form using the relations

$$x'=\frac{x}{L}, \quad z'=\frac{z}{h_0}, \quad \delta'_m=\frac{\delta_m}{h_0}, \quad t'=\frac{\sqrt{gh_0}}{L}t,$$

$$\zeta'=\frac{\zeta}{a}, \quad h'=\frac{h}{h_0}=1+\varepsilon\zeta', \quad \phi'=\frac{h_0}{aL\sqrt{gh_0}}\phi, \quad P'=\frac{P}{\rho g h_0},$$

where the primes are used to denote dimensionless quantities. Omitting the primes for the sake of clarity, the vertical momentum equation, in dimensionless form, writes

$$\varepsilon\Gamma=-1-\partial_z P,$$

where $\Gamma=\partial_t w+\varepsilon u\partial_x w+\frac{\varepsilon}{\mu}w\partial_z w$ is the vertical acceleration, $u=\partial_x\phi$ the horizontal velocity and $w=\partial_z\phi$ the vertical velocity. Integrating this equation over $z$ we get

$$\zeta=\zeta_{\mathrm{H}}-\int_{-1+\delta_m}^{\varepsilon\zeta}\Gamma dz \tag{4}$$

where $\zeta_{\mathrm{H}}$ is the dimensionless hydrostatic reconstruction

$$\zeta_{\mathrm{H}}=\frac{1}{\varepsilon}(P_{\mathrm{m}}-P_{\mathrm{atm}}-1+\delta_m). \tag{5}$$

The Formula (4) is exact but involves quantities that cannot be expressed in terms of the measured pressure $P_{\mathrm{m}}$. Our goal is to derive approximate formulas that can be expressed as a function of $P_{\mathrm{m}}$, or equivalently $\zeta_{\mathrm{H}}$. Following (Lannes and Bonneton, 2009), we shall perform an asymptotic expansion of (4) in terms of the shallowness parameter $\mu$.

The velocity potential $\phi$ is given at second order by

$$\phi=\psi-\frac{\mu}{2}\left((z+1)^2-h^2\right)\partial_x^2\psi+O(\mu^2), \tag{6}$$

with $\psi=\phi_{|_{z=\varepsilon\zeta}}$. From this equation we can deduce that

$$u=U+O(\mu)$$

$$w=-\mu(z+1)\partial_x U+O(\mu^2),$$