Probabilistic Engineering Mechanics 43 (2016) 5–19

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Models for space–time random functions

Mircea Grigoriu

Cornell University, Ithaca, NY 14853-3501, USA

ARTICLE INFO

Available online 22 November 2015

Keywords:
Karhunen–Loève (KL) representation
Random field
Singular value decomposition
Space–time random function
Stochastic process

ABSTRACT

Models are developed for random functions $Q(x, t)$ of space $x \in D$ and time $t \in [0, \tau]$ from samples of these functions and any other information when available. Most of the models in the paper can be viewed as extensions of Karhunen–Loève (KL) representations for random fields. Their samples are linear forms of basis functions with random coefficients which are extracted from samples of $Q(x, t)$ by singular value decomposition. The coefficients of these forms are stochastic processes rather than random variables. The proposed models can be used to generate large sets of samples whose statistics are similar to those of target random functions. Theoretical arguments and numerical examples are presented to establish properties of the proposed models, assess their accuracy, and illustrate their implementation.

1. Introduction

Random functions $Q(x, t)$ of space $x \in D$ and time $t \in [0, \tau]$ abound in applications, e.g., temperatures and ozone levels in cities during a day or season, pressure fields on aircrafts, buildings, and other physical systems, wave forces on offshore platforms, ships, and dikes, and pollution concentration in rivers. These functions can also be used to describe the spatial variation of material properties at small scale and their evolution in time caused by large loads and/or excessive temperatures.

Generally, the number of available records of space–time random functions $Q(x, t)$ is insufficient to estimate their probability laws and find statistics of system responses subjected to actions described by these types of random functions. For example, available records are insufficient to estimate ozone levels and wind speeds of relatively large return periods. Statistics of stress fields in aircrafts during takeoff and landing, which are essential for developing economical and safe designs, cannot be derived from a few records of random actions on these systems. Models $Q_m(x, t)$, which are consistent with the available information on target random functions $Q(x, t)$, need to be used to characterize these functions beyond their records and solve practical problems.

There is a vast literature on space–time random functions in the statistical literature, where they are referred to as spatial random processes, functional time series, and space–time random functions [13–15]. The objective of most of these studies is to construct models that match the first two moments of target space–time random functions. In contrast, our objective is to construct models whose samples are similar to those of target space–time random functions $Q(x, t)$ from the available information, which consists of samples of $Q(x, t)$ or samples of $Q(x, t)$ and some information on the probability law of this random function.

Most of the models in our discussion can be viewed as extensions of the Karhunen–Loève (KL) representations for random fields $Q(x)$, $x \in D$, i.e., random functions of space coordinates. These representations are superpositions of eigenfunctions of the correlation function of $Q(x)$ with random coefficients $\{Z_k\}$. The proposed models have the same functional form except for that stochastic processes $\{Z_k(t)\}$ are used in place of the random coefficient $\{Z_k\}$. These types of models have been used to solve stochastic partial differential equations (SPDEs) in both applied [21] and theoretical [2] (Section 3.1.1) studies. In these context, the model components are solutions of differential equations. In our context, the model components are inferred from the available information on $Q(x, t)$.

The following section restates the objective of this study in formal terms. Sections 3 and 4 develop models for time-stationary and arbitrary random functions $Q(x, t)$, present properties of these models, and develop bounds on the discrepancy between target functions $Q(x, t)$ and models of these functions. Numerical examples are in Section 5. They illustrate the construction of the proposed models for various space–time random functions $Q(x, t)$ and levels of information on these functions. The model performance can be assessed precisely since the probability laws of the target functions are known. It is found that the proposed model are satisfactory in all cases examined in this study.

2. Problem definition

Let $Q(x, t)$, $x \in D$, $t \in [0, \tau]$, be an $\mathbb{R}^q$ – valued random function of space x and time t, where D is a bounded subset of $\mathbb{R}^p$, $p = 1, 2, 3$, and $[0, \tau]$ denotes a bounded time interval. Our objective is to construct a family of models $\{Q_m(x, t)\}$, $m = 1, 2, \ldots,$

http://dx.doi.org/10.1016/j.probengmech.2015.11.004
0266-8920/

for $Q(x, t)$ from (1) a set $\{q_i(x, t)\}$, $i=1,\ldots,N$, of $N \geq 1$ independent samples of $Q(x, t)$ or (2) a set $\{q_i(x, t)\}$, $i=1,\ldots,N$, of $N \geq 1$ independent samples of $Q(x, t)$ and some properties of this random function, e.g., the knowledge that $Q(x, t)$ is a time-stationary random function.

Consider the family of random fields $\{Q(x, t), x \in D\}$ indexed by $t \in [0, \tau]$. The members of this family have the same properties if $Q(x, t)$ is time-stationary but their properties change in time if $Q(x, t)$ is not time-stationary. This observation is used to construct models for this random function. If $Q(x, t)$ is time-stationary, the properties of the random fields $\{Q(x, t), x \in D\}$ are time-invariant and can be estimated from the values of the samples $\{q_i(x, t)\}$, $i=1,\ldots,N$, at a single time $t \in [0, \tau]$. This estimation approach uses partially the available information. It can be improved by using values of $\{q_i(x, t)\}$, $i=1,\ldots,N$, at several times $t = t_r$, $r = 1, \ldots, n_r$, which are spaced adequately such that the dependence between the random fields $Q(x, t_r)$ and $Q(x, t_{r'})$, $t_r \neq t_{r'}$ is weak. If $Q(x, t)$ is not time-stationary, the law of the random fields $\{Q(x, t), x \in D\}$ changes in time so that the spatial variation of $Q(x, t)$ cannot be inferred from values of the samples $\{q_i(x, t)\}$, $i=1,\ldots,N$, at a single time.

The following two sections develop models for time-stationary and arbitrary random functions $Q(x, t)$. Theoretical arguments and numerical examples are presented in these sections to clarify features of the models of $Q(x, t)$ and illustrate their implementation. It is shown that space–time random functions can be represented by linear forms of basis functions of spatial argument $x \in D$ with random coefficients, which are stochastic processes rather than random variables. Both the basis functions and the random coefficients can be extracted from samples of $Q(x, t)$. For simplicity, it is assumed that the random function $Q(x, t)$ is real-valued with mean zero.

3. Models for time-stationary random functions

Denote by $Q(x)$, $x \in D$, an arbitrary member of the family of random fields $\{Q(x, t), x \in D\}$ indexed by $t \in [0, \tau]$. If the random field $Q(x)$, $x \in D$, has continuous square integrable correlation function $r(x, y) = E[Q(x)Q(y)]$, it admits the Karhunen–Loève (KL) representation $Q(x) = \sum_{k=1}^{\infty} Z_k\, \varphi_k(x)$, $x \in D$, where $\{Z_k\}$ are uncorrelated random variables with means 0 and variances $\{\lambda_k\}$ and $\{\lambda_k, \varphi_k(x)\}$ are the eigenvalues and eigenfunctions of the integral equation $\int_D r(x, y)\varphi(y)dy = \lambda\, \varphi(x)$, $x \in D$ [11] (Section 3.6.5). It is common in applications to approximate $Q(x)$ by truncated KL expansions, i.e., models of the type

$$Q(x) \simeq Q_m(x) = \sum_{k=1}^{m} Z_k\, \varphi_k(x), \quad x \in D,\ m = 1, 2, \ldots, \tag{1}$$

where m denotes the truncation level.

This fact suggests to describe the space–time random function $Q(x, t)$ considered in this section by versions of KL representations which allow the random variables $\{Z_k\}$ to fluctuate in time. This extension of the model in Eq. (1) has the form

$$Q(x, t) \simeq Q_m(x, t) = \sum_{k=1}^{m} Z_k(t)\varphi_k(x), \quad x \in D,\ t \in [0, \tau],\quad m = 1, 2, \ldots, \tag{2}$$

where $\{\varphi_k: D \to \mathbb{R}\}$ are deterministic functions, referred to as basis functions, and $\{Z_k(t)\}$ are real-valued, stationary stochastic processes with mean zero and finite variance, rather than random variables as in Eq. (1). The model $Q_m(x, t)$ of $Q(x, t)$ has been used in [2] (Section 3.1.1) to solve partial differential equations with space–time white noise input.

We note that the models $Q_m(x, t)$ in Eq. (2) and the target random function $Q(x, t)$ have similar properties in the sense clarified by the following property.

Property 1. The random function $Q_m(x, t)$ in Eq. (2) has mean zero and is time-stationary.

Proof. The first two moments of $Q_m(x, t)$ are $E[Q_m(x, t)] = 0$ and $E[Q_m(x, t)Q_m(y, s)] = \sum_{k,l=1}^{m} E[Z_k(t)Z_l(s)]\, \varphi_k(x)\varphi_l(y)$. The latter moment simplifies to $E[Q_m(x, t)Q_m(y, s)] = \sum_{k=1}^{m} E[Z_k(t)Z_k(s)]\, \varphi_k(x)\varphi_k(y)$ if the processes $\{Z_k(t)\}$ are uncorrelated. Since the expectations $E[Z_k(t)Z_l(s)]$ depend only on the time lag $(t-s)$, the random functions $Q_m(x, t)$ are weakly time-stationary for both correlated and uncorrelated $\{Z_k(t)\}$. Since the processes $\{Z_k(t)\}$ are stationary by assumption, we have

$$\begin{aligned} &P\big(Q_m(x, t_1) \leq \xi_1, \ldots, Q_m(x, t_d) \leq \xi_d\big) \\ &= P\left(\sum_{k=1}^{m} Z_k(t_1)\varphi_k(x) \leq \xi_1, \ldots, \sum_{k=1}^{m} Z_k(t_d)\varphi_k(x) \leq \xi_d\right) \\ &= P\left(\sum_{k=1}^{m} Z_k(t_1+s)\varphi_k(x) \leq \xi_1, \ldots, \sum_{k=1}^{m} Z_k(t_d+s)\varphi_k(x) \leq \xi_d\right) \\ &= P\big(Q_m(x, t_1+s) \leq \xi_1, \ldots, Q_m(x, t_d+s) \leq \xi_d\big) \end{aligned}$$

for arbitrary $d \geq 1$ and times $t_1, \ldots, t_d, s$, so that the random functions $Q_m(x, t)$ are time-stationary.□

The implementation of the model in Eq. (2) requires to select the basis functions $\{\varphi_k\}$ and specify the properties of the stochastic processes $\{Z_k(t)\}$. We accomplish this task in two steps. First, the basis functions $\{\varphi_k\}$ are constructed from the spatial variation of the samples of $Q(x, t)$ at arbitrary times. Second, the orthogonality property of the basis functions is used to extract samples of $\{Z_k(t)\}$ from samples of $Q(x, t)$, which are subsequently used to construct models for these processes.

3.1. Spatial variation

Let $Q(x)$, $x \in D$, continue to denote $Q(x, t)$ at an arbitrary time $t \in [0, \tau]$. Let $\mathcal{H}$ be a Hilbert space with norm induced by an inner product $\langle \cdot, \cdot \rangle$ that contains the samples of $Q(x)$ and let $\{e_i\}$, $i \in I$, be an orthonormal basis of $\mathcal{H}$, where I is an index set. An arbitrary element e of $\mathcal{H}$ admits the representation $e = \sum_{i \in I} \langle e, e_i \rangle\, e_i$, where $\{\langle e, e_i \rangle\}$ are Fourier coefficients. It can be shown [12] (Theorem 3.4.10) that the set of non-zero Fourier coefficients is countable, so that $e = \sum_{k=1}^{\infty} \langle e, e_k \rangle\, e_k$. Accordingly, the samples $Q(x, \omega)$ of $Q(x)$ admit the representation

$$Q(x, \omega) = \sum_{k=1}^{\infty} Z_k(\omega)\varphi_k(x), \quad x \in D, \tag{3}$$

where the functions $\{\varphi_k(x)\}$ define an orthonormal basis, i.e., $\int_D \varphi_k(x)\varphi_k(x)dx = \delta_{kl}$, and $Z_k(\omega) = \langle Q(\cdot, \omega), \varphi_k(\cdot) \rangle = \int_Q Q(x, \omega)\varphi_k(x)dx$ are Fourier coefficients.

Truncated versions,

$$Q_m(x, \omega) = \sum_{k=1}^{m} Z_k(\omega)\varphi_k(x), \quad x \in D, \tag{4}$$

of the representation in Eq. (3) have the form of the model of $Q(x, t)$ in Eq. (2) for an arbitrary time. The representation of $Q(x)$ in Eq. (4) is an element of the subspace $\mathcal{H}_m$ of $\mathcal{H}$ spanned by the basis functions in this equation, i.e., $\mathcal{H}_m = \mathrm{span}(\varphi_1, \ldots, \varphi_m)$, and represents the projection of $Q(x)$ on $\mathcal{H}_m$.

The error field $\varepsilon_m(x) = Q(x) - Q_m(x)$ with samples

$$\varepsilon_m(x, \omega) = Q(x, \omega) - Q_m(x, \omega) = \sum_{k=m+1}^{\infty} Z_k(\omega)\varphi_k(x), \quad x \in D. \tag{5}$$