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Adaptive estimation of statistical moments of the responses of random systems

Wenliang Fan [a,b,c,*], Jinghong Wei [b], Alfredo H-S Ang [c], Zhengliang Li [a,b]

[a] Key Laboratory of New Technology for Construction of Cities in Mountain Area(Chongqing University), Ministry of Education, Chongqing 400045, China
[b] School of Civil Engineering, Chongqing University, Chongqing 400045, China
[c] Department of Civil and Environmental Engineering, University of California, Irvine, CA 92697, USA

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ABSTRACT

Estimation of statistical moments of structural response is one of the main topics for the analysis of random systems; balancing between accuracy and efficiency is still a challenge. In this work, a new point estimate method (PEM) based on adaptive dimensional decomposition is presented. Firstly, by introducing the normal-to-nonnormal transformation, a system with general variables is changed into one with independent standard normal variables. Secondly, the criterion for delineating the existence of cross terms is derived theoretically, together with its practical implementation. Thirdly, by combining with the dimension-reduction method (DRM), an adaptive dimensional decomposition for moment function is proposed and the PEM based on adaptive dimensional decomposition, which comprises two types of sub-methods, is developed. Finally, several examples are investigated to verify the accuracy, efficiency, convergence and rationale of the proposed method.

1. Introduction

Evaluation of the statistical moments of structural response is one of the main topics for the analysis of random system. In theory, this involves the high dimensional integral of the joint probability density function (PDF). The point estimate method (PEM) of the moments of the system response is an efficient approach, in which deterministic structural analysis is necessary.

In general, point estimate method can be classified into two types. In the first type, the integrand of the high dimensional integral involves the exact response function; for the second type, the response function is replaced by its approximate integrand.

The earliest research for the first type came from Rosenblueth [1,2], of which 2^n re-analysis are necessary for a structure with n random variables which will be expensive for large n. By introducing the eigensystem of the correlation matrix of the basic variables, Harr extended Rosenblueth's PEM to accommodate large number of random variables, where only $2n$ points are required [3]. Chang et al. proposed a modified Harr's method based on the standardized eigenspace [4], and then generalized the method to systems with nonnormal random variables by introducing the Nataf transformation [5]. All the methods mentioned above have low precision for strongly nonlinear systems. Through the Rosenblatt transformation or Nataf transformation, Zhou and Novak transformed the original variables into independent standard normal variables, and then evaluated the moments by various Gauss quadrature rules [6]; however, the resulting rules are also not suitable for strongly nonlinear systems.

In the second type for estimating the moments, there are usually two kinds of approximations of multivariable functions. One is the product approximation, the other is the summation approximation. Most studies have concentrated on the summation approximation, in which the multivariate function is approximated by the sum of low-dimension functions. Based on the Taylor expansion of a multivariate function, Li approximated a general function by the summation of univariate quartic polynomials and the 2nd cross terms of all the variables, in which the expectation of univariate functions are evaluated by a three-point method and the coefficients of the 2nd cross terms are approximated by finite differences [7]. Zoppou and Li and Tsai and Franceschini replaced the univariate quartic polynomials in Li's method by univariate linear polynomials and cubic polynomials, respectively [8,9]. Hong extended PEM for univariate function to multivariate function directly, with a multivariate function in approximation [10]. Rahman and Xu proposed similar summation

* Corresponding author. Tel.: +86-15178817816.
E-mail address: davidfwl@cqu.edu.cn (W. Fan).

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approximation, namely a univariate dimension-reduction method (DRM) [11], together with its rigorous proof, and then develop a generalized DRM in which a multivariate function is approximated by the summation of low-dimension functions [12]. However, all the methods above are not convenient for strongly nonlinear systems and for higher probabilistic moments due to the difficulty in determining the points. For the standard normal variable, the points and associated weights is determined easily by the nodes and weights of Gauss–Hermite quadrature, hence the transformations between arbitrary random vector and independent standard normal vector is very useful for estimating moments. Zhao and Ono presents an efficient and easy-to-implement PEM [13], which combines the univariate DRM with the Rosenblatt transformation in essence, and Li et al. develops a more practical PEM by substituting the Rosenblatt transformation with the Nataf transformation [14], and Zhang et al. further combines the bivariate DRM with the Rosenblatt transformation of an dependent random vector [15].

The DRM has become a promising method for estimating the moments. Even if the multivariate DRM is more accurate and theoretically appropriate, the univariate DRM is more popular because of its high efficiency. Therefore, improving the efficiency of multivariate DRM is essential and useful.

This paper presents an adaptive multivariate dimension-reduction model for multivariate functions based on the delineation of the cross terms, thus deleting the unnecessary terms of the traditional multivariate dimension-reduction model. It is organized as follows. In Section 2 the theoretical basis for delineating the existence of the cross terms is proposed, together with its practical implementation. Then, by combining the remodeling of a multivariate function based on normal-to-nonnormal transformation, the dimensional decomposition of a general function and delineating the existence of cross terms, an adaptive dimensional decomposition for a general function and its higher order moment functions is presented in Section 3, including the point estimates of the pertinent moments based on the adaptive dimensional decomposition. To verify the accuracy, efficiency, convergence and rationale of the proposed point estimated methods, several mathematical examples and engineering cases are numerically examined in Section 4. Finally, Section 5 summaries some conclusions.

2. Delineating the existence of cross terms

In a random system, its stochastic response can be expressed by

$$Z = g(\boldsymbol{\Theta}) \tag{1}$$

where Z is the system response, $\boldsymbol{\Theta}=(\Theta_1,\Theta_2,\ldots,\Theta_n)^T$ is the basic random vector of the system, and $g(\bullet)$ is a general function describing the relation between the input vector $\boldsymbol{\Theta}$ and the output Z. And the interactive relation of Θ_i and Θ_j on Z is described by the cross terms of Θ_i and Θ_j in $g(\bullet)$. However, the impact of the components of $\boldsymbol{\Theta}$ can be independent or correlated. Therefore, there are cross terms of Θ_i and Θ_j in Eq. (1) if the interactive relation of Θ_i and Θ_j on Z exists, or else the cross terms of Θ_i and Θ_j disappear.

2.1. Theoretical background

Lemma 1. *If $\boldsymbol{\Theta}=(\Theta_1,\Theta_2)^T$, the sufficient and necessary condition for $g(\Theta_1,\Theta_2)$ being without the cross terms of Θ_1 and Θ_2 is that*

$$\left[g\left(\theta_1^{(2)},\theta_2^{(1)}\right)-g\left(\theta_1^{(1)},\theta_2^{(1)}\right)\right]-\left[g\left(\theta_1^{(2)},\theta_2^{(2)}\right)-g\left(\theta_1^{(1)},\theta_2^{(2)}\right)\right]=0 \tag{2}$$

for all possible $\theta_i^{(k)}$, where $\theta_i^{(k)}$ (i,k=1,2) is the kth sample of Θ_i.

Proof. : According to Taylor's expansion, there exists

$$g\left(\theta_1^{(2)},\theta_2^{(1)}\right)-g\left(\theta_1^{(1)},\theta_2^{(1)}\right)=\sum_{l=1}^{\infty}\frac{1}{l!}\frac{\partial g^l\left(\theta_1^{(1)},\theta_2^{(1)}\right)}{\partial\Theta_1^l}\left(\theta_1^{(2)}-\theta_1^{(1)}\right)^l \tag{3}$$

then

$$\left[g\left(\theta_1^{(2)},\theta_2^{(1)}\right)-g\left(\theta_1^{(1)},\theta_2^{(1)}\right)\right]-\left[g\left(\theta_1^{(2)},\theta_2^{(2)}\right)-g\left(\theta_1^{(1)},\theta_2^{(2)}\right)\right]=\sum_{l=1}^{\infty}\frac{1}{l!}\left[\frac{\partial g^l\left(\theta_1^{(1)},\theta_2^{(1)}\right)}{\partial\Theta_1^l}-\frac{\partial g^l\left(\theta_1^{(1)},\theta_2^{(2)}\right)}{\partial\Theta_1^l}\right]\left(\theta_1^{(2)}-\theta_1^{(1)}\right)^l \tag{4}$$

Similarly, it is clear that

$$\frac{\partial g^l\left(\theta_1^{(1)},\theta_2^{(2)}\right)}{\partial\Theta_1^l}-\frac{\partial g^l\left(\theta_1^{(1)},\theta_2^{(1)}\right)}{\partial\Theta_1^l}=\sum_{m=1}^{\infty}\frac{1}{m!}\cdot\frac{\partial g^m}{\partial\Theta_2^m}\left[\frac{\partial g^l\left(\theta_1^{(1)},\theta_2^{(1)}\right)}{\partial\Theta_1^l}\right]\cdot\left(\theta_2^{(2)}-\theta_2^{(1)}\right)^m \tag{5}$$

Substituting Eq. (5) into Eq. (4) reads

$$\left[g\left(\theta_1^{(2)},\theta_2^{(1)}\right)-g\left(\theta_1^{(1)},\theta_2^{(1)}\right)\right]-\left[g\left(\theta_1^{(2)},\theta_2^{(2)}\right)-g\left(\theta_1^{(1)},\theta_2^{(2)}\right)\right]=-\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{l!}\frac{1}{m!}\cdot\frac{\partial g^{m+l}\left(\theta_1^{(1)},\theta_2^{(1)}\right)}{\partial\Theta_2^m\partial\Theta_1^l}\cdot\left(\theta_2^{(2)}-\theta_2^{(1)}\right)^m\left(\theta_1^{(2)}-\theta_1^{(1)}\right)^l \tag{6}$$