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Multiple timescale spectral analysis

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ABSTRACT

Spectral analysis is a classical tool for the structural analysis of structures subjected to random excitations. The most common application of spectral analysis is the determination of the steady-state second order cumulant of a linear oscillator, under the action of a stationary loading prescribed by means of its power spectral density. There exists however a broad variety of such similar problems, extending the concept to multi degree-of-freedom systems, non Gaussian excitation, slightly nonlinear oscillators or even transient excitations. In this wide class of problems, the cumulants of the response are obtained as the result of the integral of corresponding spectra over the frequency space, which is possibly multi-dimensional. Application of standard numerical integration techniques may be prohibitive, a reason why the spectral approach is often left aside. Besides, many engineering problems involve a clear timescale separation, usually of those pertaining to the loading and to the mechanical behavior of the system. In these problems, a proper consideration of the timescale separation results in dropping the order of integration by one, at least. This offers the possibility to derive analytical solutions, whenever the order of integration drops to zero, or to make numerical integration competitive. The paper presents this general method, together with some applications in wind and marine engineering.

1. Introduction

Many engineering applications are based on the input/output representation of a system, see Fig. 1(a), which is well suited to study the stochastic response of deterministic systems to random input. There is a wide variety of such applications in the various fields of engineering [1–4] among which classical problems of structural and mechanical applications such as the random vibrations of aircraft and spaceships in aerospace engineering, the vibrations of mechanical components [5], the response of civil structures to seismic [6], wind [7] or wave loading [8]. Although the concepts developed in the paper are not limited to the latter domains, the state-of-the-art references are mainly borrowed from structural dynamics, which also serves as the major field of application for the illustrations given in the paper. In many of these applications the timescales $\{T^\star\}$ associated with the system are significantly different from those $\{t^\star\}$ of the loading (input) so that the response turns out to be a composition of components with different timescales. The cases where $\{t^\star\} \ll \{T^\star\}$ or $\{T^\star\} \ll \{t^\star\}$ are the ground motivation for this paper.

Traditional simulation techniques are rather inefficient in solving these kinds of problems as they would have to simultaneously resolve the different scales, which translates in very long simulations with very short timesteps [9,10].

Alternatively stochastic spectral analysis appears as an appealing tool to solve these problems. Indeed, the loading is usually given with a spectral representation, *i.e.* a set of spectra defined on multidimensional frequency spaces. Among them, the well-known power spectral density, defined for $\omega \in \mathbb{R}$, represents the distribution of the second cumulant (the variance) over the frequency domain. More generally, the jth-order spectrum represents the distribution of the jth stationary cumulant over a frequency domain $\mathbb{R}^{j-1}$. The objective of a spectral analysis is to determine the spectra of the response in terms of those of the input and eventually integrate them in the corresponding frequency spaces in order to determine the cumulants of the response. Spectral analysis develops in various versions: (i) the Gaussian and the non-Gaussian versions, *i.e.* limited to statistical order 2 or not, e.g. [11–15], (ii) the single degree-of-freedom (SDOF) and the multiple degree-of-freedom (MDOF) versions, the latter being well known for the second order [5,7] and theoretically established for higher order analysis [13,16], (iii) the steady-state response analysis, as in most common applications, but also with evolutionary stochastic analyses, a concept formalized by Priestley [17], but that still receives much interest [18], especially in seismic engineering [19,20] and (iv) although the concept of spectral analysis hinges on the underlying assumptions of linearity and of the superposition principle, both the stochastic linearization [21] (as well as its higher order generalizations [22–24]) and the Volterra series

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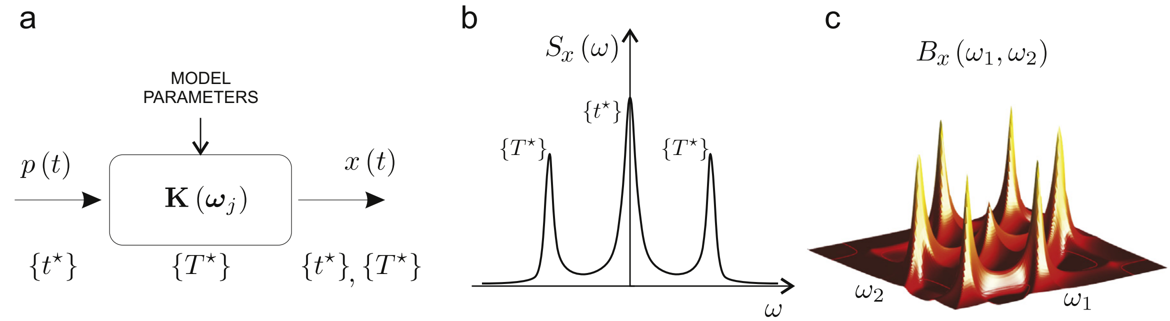

Fig. 1. (a) Input/output representation of a system; (b and c) examples of kernels $\mathbf{K}(\boldsymbol{\omega}_j)$ encountered in spectral analysis: second- and third-order stationary responses.

model [25–27] are interesting ways to extend its domain of applicability to systems with mild nonlinearities. In all these versions of the spectral analysis, a canonical form for the *j*th order spectrum of the response (or contribution thereof in the nonlinear case) is

$$\mathbf{S_x}(\boldsymbol{\omega}_j) = \mathbf{K}(\boldsymbol{\omega}_j)\mathbf{S_p}(\boldsymbol{\omega}_j) \tag{1}$$

where $\boldsymbol{\omega}_j = \{\omega_1, \ldots, \omega_{j-1}\}$ with $j \geq 2$ gathers the independent variables of the frequency space, $\mathbf{K}(\boldsymbol{\omega}_j)$ is a frequency kernel function and $\mathbf{S_p}(\boldsymbol{\omega}_j)$ and $\mathbf{S_x}(\boldsymbol{\omega}_j)$ respectively stand for the *j*th-order spectra of the loading and of the response. A modified version, adding time *t* as an additional independent variable, is required in case of evolutionary spectrum [19]. The spectral analysis then consists in the determination of

$$\mathbf{k}_j = \int \cdots \int_{\mathbb{R}^{j-1}} \mathbf{S_x}(\boldsymbol{\omega}_j)\, d\boldsymbol{\omega}_j \tag{2}$$

which represents the *j*th cumulant of the response (or a contribution thereof in the nonlinear case). Examples of such kernels are sketched in Fig. 1(b and c).

The existence of multiple timescales in the response translates into the existence of several well-distinct peaks in the spectra, see Fig. 1(b and c). To handle multiple timescales in the response is thus less problematic than in a crude Monte Carlo simulation. Indeed, the spectral analysis smartly solves the possible issue with an adequate distribution of integration points necessary to determine $\mathbf{k}_j$ in (2). Additionally to those used to properly capture the loading $\mathbf{S_p}(\boldsymbol{\omega}_j)$, they have to be distributed in tiny zones around the poles of the kernel $\mathbf{K}(\boldsymbol{\omega}_j)$, *i.e.* the natural frequencies in case of a linear system.

For $j=2$, some authors avoid the difficulty to justify the truncation of the domain of integration with a suitable change of variable, mapping the infinite frequency domain $\mathbb{R}$ onto a finite interval one [28]. It seems that the concept could be extended to higher dimensional integration ($j > 2$). In other contributions, robust adaptive quadrature schemes [29,30] provide accurate results with limited computational investment, especially if the known location of the resonance peaks is considered to initiate the iterative numerical integration. Although adaptive quadrature is extended to high-dimensional integration [31,32], integration of high-dimensional spectra such as the bispectrum ($j=3$) and the trispectrum ($j=4$) has been reported to require excessively large computation times [33,34], mainly because integration in higher dimensions actually requires an exponentially growing number of integration points. Monte Carlo sampling techniques [9] could help accelerating the numerical integration in high-dimensional spaces, but it still requires a tremendous amount of integration points, even if cut by a large factor compared to adaptive quadrature. The end pursued in this paper is to provide, thanks to a timescale separation technique, simplified expressions for these integrals.

Evidences of excessively large computation times trigger the questioning on the applicability of the spectral analysis up to a large order, say fourth order, for large systems. By large systems, we mean those who require a complex finite element modeling with a large number *k* of DOFs in which case the establishment of the spectral matrix of the loading requires a significant time per integration point $\boldsymbol{\omega}_j$. This number is of order 10^6 today. The larger the size $k \times \cdots \times k$ (*j* times) of that matrix, the worse, of course. Linear systems with a large number of DOFs are analyzed in their modal basis, in which case (1) represents a modal analysis and the size of matrices $\mathbf{S_p}$ and $\mathbf{S_x}$ drops to a few tens or hundreds per dimension. The projection of the spectral representation of the loading (input) in the modal basis, *i.e.* the determination of $\mathbf{S_p}(\boldsymbol{\omega}_j)$, definitely represents the most important part of the computational burden. Indeed, this operation requires the projection of costly multidimensional nodal matrices. Further observing that this operation has to be repeated for a large number of integration points $\boldsymbol{\omega}_j$, this explains why usage of a modal basis reduction only partly solves the computational issue, and thus why only the theoretical setting of the non-Gaussian spectral analysis that was extensively studied in the mid 1990s (e.g. [11,35,15,36]) is practically limited to structures with very few DOFs ([37]). On the other hand, nonlinear systems with a large number of DOFs are usually analyzed by means of model reduction techniques or statistical linearization concepts, in which case a linear modal analysis follows, which extends the validity of the above observation.

The second-order spectral analysis is now implemented in many structural analysis softwares, be they commercial or not. Large power spectral density matrices of loading are projected in the modal space and the resulting spectra are integrated with accurate numerical methods. Although not completed with a simple click yet, these operations necessitate reasonable computational times, thanks to the resources available on today's personal computers. Even though the number of transistors on a micro-processor chip doubles every 18 months according to Moore's law, acceptable computational times for the third- and the fourth-order analysis of large systems is not to be expected in a near future, at least with a crude application of (2). Looking backward in time however reveals interesting perspectives. In 1961, driven by the need to provide an efficient way of calculating the second- order response of structures subjected to buffeting forces (what is actually rather fast today), Davenport suggested to decompose the response in two components, the background and resonant components [38]. In that approach, the number of integration points thus dropped to the sole natural frequencies of the (possibly very large) structural model. With the benefit of hindsight, we can see this method as the leading order solution of a multiple timescale approach, in which the slow timescale related to the wind loading activates the fast structural dynamic responses