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The asymptotic stochastic strength of bundles of elements exhibiting general stress–strain laws

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ABSTRACT

The fiber bundle model is widely used in probabilistic modeling of various phenomena across different engineering fields, from network analysis to earthquake statistics. In structural strength analysis, this model is an essential part of extreme value statistics that governs the left tail of the cumulative probability density function of strength. Based on previous nano-mechanical arguments, the cumulative probability distribution function of strength of each fiber constituting the bundle is assumed to exhibit a power-law left tail. Each fiber (or element) of the bundle is supposed to be subjected to the same relative displacement (parallel coupling). The constitutive equations describing various fibers are assumed to be related by a radial affinity while no restrictions are placed on their particular form. It is demonstrated that, even under these most general assumptions, the power-law left tail is preserved in the bundle and the tail exponent of the bundle is the sum of the exponents of the power-law tails of all the fibers. The results have significant implications for the statistical modeling of strength of quasibrittle structures.

1. Introduction

Since the pioneering work by Peirce [1] in the 1920s, the study of fiber bundle models has attracted an increasing attention by the research community. This is due to its proven effectiveness in the analysis of various phenomena across different engineering fields – from material science to earthquake statistics, or from network to traffic modeling, just to mention a few [2–12].

Recently, Kim et al. [2] applied the fiber bundle to study the overloading failures in power grids. The load transfer of broken fibers or nodes through the edges or links of the underlying network was assumed to be governed by the local load-sharing (LLS) rule [13–16].

The same load-sharing rule was adopted by Chakrabarti [3] for the study of (global) traffic jam in a city network of roads. In this case, for some special distributions of traffic handling capacities of the roads, the application of the fiber bundle model let the analytical study of the critical behavior of the jamming transition.

On parallel tracks, Moreno et al. [4] applied the fiber bundle model to study the complex aftershocks sequences which occur after earthquakes. Each element, an asperity or barrier, was supposed to break because of static fatigue, transferring its stress according to a local load-sharing rule and then regenerate.

In material science, the fiber bundle was extensively used as a tool for studying important phenomena such as fracture, fatigue, creep or thermally induced failure of brittle, ductile and quasibrittle materials [5–12].

Particularly important for structural strength is the asymptotic behavior of the fiber bundle because it governs the extreme value statistics of strength of engineering structures. This is essential for safe design of aircraft, bridges, dams, nuclear structures and ships, as well as microelectronic components and medical implants, since the tolerable failure probability is extremely low, typically $P_f \leq 10^{-6}$.

An early rigorous study of the cumulative probability distribution function (cdf) of the strength and of some asymptotic properties of a fiber bundle with brittle elements (called fibers) was carried out by Daniels [17]. He derived an exact recursive formula for computing the cdf regardless of the particular type of cdf of individual fibers. He also demonstrated that, for the limit case in which the number of fibers approaches infinity, the cdf of strength tends to the Gaussian (or normal) distribution regardless of the cdf of individual brittle fibers. Later, this property was also demonstrated by different approaches by Galambos [18], Smith [19], Sornette [20] and Le et al. [21].

The left tail of the strength distribution for bundles of brittle fibers was studied by Harlow et al. [22] within the framework of set theory. In their work, it was proven in detail that the left tail of the cdf of strength of a fiber bundle is a power-law tail if the elements exhibit a left power-law tail, although this property was already implied in the pioneering work of Fisher and Tippet [23],

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in terms of the stability postulate of extreme value statistics. Furthermore, Harlow et al. showed that the tail exponents of brittle fibers are additive, i.e., the exponent of the bundle tail is the sum of the exponents of the power-law tails of all its brittle fibers.

For brittle fibers, the same result was derived by Bažant and Pang [24] in a simpler way using the recursive formula proposed by Daniels [17]. Further, in their paper, the authors demonstrated the presence of a power-law left tail and the additivity of the exponents for ductile fibers. Then they coupled the fiber bundle model to the weakest-link model to describe the multi-scale transition of strength from the nanoscale to the macroscale. They also pioneered the analysis of the reach of power-law tail and showed the reach decreases by one order of magnitude for each element in the bundle. A similar behavior was later demonstrated by Le et al. [21] for fibers characterized by a bilinear stress–strain curve with a gradual post-peak softening.

All the previous analyses of the asymptotic behavior of bundle strength relied on the assumption of a particular constitutive behavior of its elements. In the present work, the constitutive equations describing each fiber are merely assumed to be related by radial affinity while no assumptions are made on their particular form. The cdf of strength of each fiber constituting the bundle is assumed to exhibit a power-law left tail.

It is demonstrated that the power-law left tail is preserved in the bundle. Moreover, it is shown that the exponent is the sum of the exponents of the power-law of each fiber.

2. Preliminary considerations

In the bundle model, after one element fails, the load gets redistributed among the other elements. The total load reduces to zero when all the elements break but the maximum load is reached when only a certain fraction of the elements is subjected to the failure load. The load redistribution after a fiber breaks depends on the load-sharing rule. Various rules have been assumed in the literature, such as the load-sharing by the nearest neighbors of the failing element in the bundle [13–17,25,26].

In the present work, all the fibers are supposed to be subjected to the same relative displacement (note that the more general case of proportional displacements can be transformed to this case). Accordingly, the load-sharing rule is fully determined by the constitutive behavior of the fibers.

The constitutive equations, $s_i = f_i(\varepsilon_i)$, describing the stress–strain law of each ith fiber are supposed to be related by the following affinity:

$$s_0 = \sigma_0 f(\varepsilon) \tag{1}$$

$$s_i = \sigma_i f\left(\beta_i \frac{\sigma_0}{\sigma_i}\varepsilon\right) \quad (i = 1 \ldots n_b) \tag{2}$$

Accordingly, Eq. (1) represents a reference curve while Eq. (2) describes the behavior of each fiber. The parameter σ_i is the strength of the ith fiber, β_i is a positive constant and n_b is the total number of fibers in the bundle. Also, $f(\varepsilon) \in [0,1]$ and it is assumed to have piecewise C^1-continuity in the domain of interest.

It is worth noting that when $\beta_i = 1$, Eqs. (1) and (2) provide a family of curves related by a radial scaling transformation. In case $\beta_i = \sigma_i/\sigma_0$, one gets a vertical scaling. A similar transformation was successfully applied within the framework of the microplane model for the scaling of functions characterizing the constitutive properties of quasibrittle materials [27]. As an example, Fig. 1 shows a family of $\sigma - \varepsilon$ curves describing a material exhibiting a post-peak softening behavior.

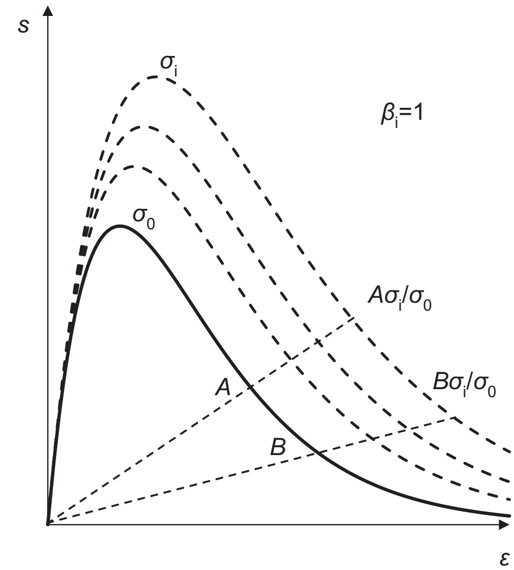

Fig. 1. Example of radially affine stress–strain curves described by Eqs. (1) and (2).

Based on the present assumptions, once the reference curve is defined, each fiber is fully characterized by its strength, σ_i, and its strain at peak, ε_i^*. Note also that no assumptions are made on the particular stress–strain behavior of each fiber. This will guarantee the generality of the results demonstrated in the following sections.

3. Analysis of the left tail of the strength cdf of the fiber bundle

Let us study the tail of the cdf of strength of the fiber bundle model. In contrast to [13–17,25,26], the fibers are supposed to be subjected to the same relative displacement while the load is redistributed within the bundle according to the stress–strain law of each fiber. A surprisingly simple property is demonstrated for power-law tails. The power-laws are always preserved and their exponents are additive. Specifically, if the cdf of strength of each of n_b fibers in a bundle has a power-law tail of exponent p_i ($i = 1 \ldots n_b$), then the cdf of bundle strength has also a power-law tail and its exponent is $p = \sum_{i=1}^{n_b} p_i$.

For a brittle bundle, this remarkable property was proven by induction based on the set theory [22,28]. Later it was also proven more simply by Bažant and Pang [24,29] by means of asymptotic expansion of Daniel's [17] exact recursive equation for the strength of cdf of bundles of increasing n_b. For a plastic bundle, this property was simply proven by induction using the joint probability theorem [24,29]. However, all the previous demonstrations relied on the assumption of a particular stress–strain curve of each element.

For the broad case of fibers with general constitutive behaviors, an analytical proof of the tail exponent additivity has been lacking. It is presented next assuming that each stress–strain curve can be described by an affinity transformation.

3.1. Analytical demonstration

For convenience, the analysis will be initially restricted to two fibers. Then, the demonstration will be extended to the general case of a fiber bundle constituted by n_b fibers.

Consider two fibers and let them be numbered such that $\sigma_1 \le \sigma_2$. Let the only random variables be the peak strengths σ_i (i=1,2) and the ratio σ_1/σ_2 while the reference strength σ_0 and