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# Uncertainty of the fatigue damage arising from a stochastic process with multiple frequency modes

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A R T I C L E I N F O

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A B S T R A C T

Predicting the variance of the fatigue damage due to a stochastic load process is a difficult classical problem that dates back to the 1960s. For many years, the available analytical methods for tackling this problem have been limited to the linear oscillator response under Gaussian white noise excitation. In a recent prior work, the author developed an improved method for calculating the damage variance for a general narrowband Gaussian process. From a fatigue uncertainty perspective, a narrowband process is particularly crucial as the amplitude correlation magnifies the variance. This paper extends the analysis to a multimodal process comprising two or more narrowband components. The proposed method is tractable, involving a single summation for an arbitrary spectral density. Moreover, closed form expressions are available for two special cases, i.e. the components are all linear oscillator responses or the spectral density of each is rectangular. The equations also yield insight on the multilayered correlation mechanisms produced by different narrowband components. The accuracy of the method is verified by rainflow counting of simulated time history stresses.



## 1. Introduction

Fatigue life prediction is an integral aspect of design for many engineering structures and mechanical components. One problem that plagues designers is that fatigue assessment is rife with high levels of uncertainties. There are parameter uncertainties; for instance nominally identical specimens under laboratory tests manifest significant scatter in the number of cycles to failure. Model uncertainties also abound from the use of damage accumulation models and the divergence of actual loading conditions from ideal experimental setups. The various uncertainties can be taken into account through probabilistic design [1], or more simply by adopting partial safety factors.

This paper is concerned with the uncertainty of the predicted fatigue life as a result of the load being characterized as a stochastic process. A stationary stochastic process may be represented by its power spectrum, and spectral fatigue analysis is the process of calculating the fatigue damage directly from the spectrum, without recourse to simulation. For a narrowband Gaussian process, an exact formula is available for the mean damage [2]. Wideband processes are more complicated, and there is a wide array of approaches for conducting the spectral fatigue analysis (see Ref. [3] for a review), although all of the available approaches are approximate.

The variance is a fundamental measure of uncertainty. The interest in the variance of the fatigue damage dates back to the 1960s, when Crandall et al. [4] and Bendat [5] separately investigated techniques for estimating the variance. Subsequently, Shinozuka [6] and Winterstein [7] also contributed to this subject. Nevertheless, the methods established by the aforementioned researchers [4–7] are applicable only to the linear oscillator response to Gaussian white noise excitation. This limitation is a reflection of the difficulty of the fatigue variance problem, as noted by Lin [8] in his classic text on stochastic dynamics. Recently, Low [9] developed an accurate method for analyzing the variance. This method requires a single summation for an arbitrary narrowband Gaussian process, and closed form results are accessible for special cases.

From the standpoint of the fatigue damage uncertainty, narrowband processes are both critical and challenging. Critical, because the strong correlation between neighboring cycle amplitudes magnifies the variance considerably. For the same reason, the need to quantify the correlation effects makes the analysis challenging. A closely related process often arising in engineering is a multimodal process comprising more than one frequency mode, each of which is narrowband. An example is the resonant response of a multi-degree-of-freedom system that is lightly damped. Multimodal processes inherit many of the narrowband traits, albeit in a more sophisticated form. The abovementioned

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correlation effects become more intricate, as they occur on several layers, with interaction between the various layers.

There are plenty of studies devoted to the mean fatigue damage for bimodal or multimodal processes [10–15], but to date there has been no attempt to investigate the variance. As such, this paper aims to shed some light on this little known subject, and to develop an analytical approach for predicting the variance of the fatigue damage pertaining to multimodal processes. It is important to clarify that the uncertainty is presumed to originate exclusively from the stochastic loading. Other sources of uncertainties are not explicitly considered in this work, but they can be included in design by other existing methods. Finally, it is worth mentioning that multimodal processes are generally more demanding to simulate. The simulation duration must be lengthy to contain sufficient cycles of the lowest frequency mode, and yet the time step should be amply fine to accommodate the highest mode. In this respect, the availability of a fast analytical approach becomes especially appealing.

## 2. Fatigue analysis of narrowband processes

### 2.1. Fundamentals of stochastic processes

Let $X(t)$ represent a stationary Gaussian process, whose autocorrelation function is written as

$$R_{XX}(\tau)=E[X(t)X(t+\tau)] \tag{1}$$

where $\tau$ is a separation time. Further, define an autocorrelation coefficient

$$\rho_{XX}(\tau)=\frac{R_{XX}(\tau)-\overline{X}^2}{\sigma_X^2}, \tag{2}$$

by normalizing $R_{XX}(\tau)$ with respect to the mean $\overline{X}$ and variance $\sigma_X^2$. The single-sided spectral density $S_{XX}(\omega)$ is obtained by taking the Fourier transform of $R_{XX}(\tau)$ [16]. The $i$th spectral moment $q_i$ is given by

$$q_i=\int_0^\infty \omega^i S_{XX}(\omega)d\omega, \quad i=0,\ 1,\ 2,\ldots \tag{3}$$

For even values of $i$, $q_i=\text{var}[\text{d}^{i/2}X(t)/\text{d}t^{i/2}]$; i.e. $q_0=\sigma_X^2$, $q_2=\sigma_{\dot{X}}^2$, etc. The spectral moments are also useful for characterizing the bandwidth. A common bandwidth parameter has the form [16]

$$\delta=\sqrt{1-\frac{q_1^2}{q_0 q_2}}, \quad 0<\delta<1. \tag{4}$$

For a narrowband process, $\delta$ approaches zero, whereas for a wideband process, $\delta \gg 0$.

A narrowband process is akin to a sequence of sinusoidal waves with modulating amplitude $A(t)$ and phase $\theta(t)$. Accordingly, a narrowband process is often modeled as

$$X(t)=A(t)\cos\,[\omega_c t+\theta(t)t]. \tag{5}$$

Here, $\omega_c=2\pi v_0^+$ is a characteristic frequency (in rad/s), where $v_0^+=\sigma_{\dot{X}}/2\pi\sigma_X$ is the zero upcrossing rate for a Gaussian process [16]. For a Gaussian process that is very narrowband, the probability density function (pdf) of the peaks $P$ follows the Rayleigh distribution:

$$f_P(p)=\frac{p}{\sigma_X^2}\exp\left(-\frac{p^2}{2\sigma_X^2}\right), \quad p\geq 0. \tag{6}$$

The amplitude $A(t)$ may be regarded as an envelope process. The Cramer and Leadbetter [17] envelope is defined as

$$A(t)=\sqrt{X^2(t)+Y^2(t)} \tag{7}$$

where $Y(t)$ is the Hilbert transform of $X(t)$. The processes $X(t)$ and $Y(t)$ have identical statistical properties, but they are uncorrelated for a given $t$.

### 2.2. Mean fatigue damage

Fatigue design is often based on the $S$–$N$ curve, which relates the stress range $S$ to the number of cycles to failure $N_f$. Usually, the portion of the $S$–$N$ curve of interest can be suitably represented by the form $N_f(S)=KS^{-m}$. The parameters $K$ and $m$ are obtained empirically from cyclic load tests of specimens. The $S$–$N$ curve has considerable scatter, which is typically embodied in the parameter $K$. For example, Wirsching and Chen [1] suggested a coefficient of variation (CoV) of 0.56 for $K$. In this work, $K$ and $m$ are assumed to be constants.

The $S$–$N$ data is directly applicable for fatigue life prediction only when $S$ is constant. To aggregate the effects of different $S$, it is necessary to invoke a damage accumulation law, the most popular of which is Miner's rule. With reference to Fig. 1, assume that each peak or valley (of magnitude $P$, with $P\geq 0$) represents half a fatigue cycle. The reason for considering half cycles, as opposed to full cycles, is to facilitate the subsequent treatment of the variance. The stress range $S$ is two times a peak or valley, i.e. $S=2P$. Miner's rule stipulates that the fatigue damage accumulates linearly, such that the total damage due to $N$ half-cycles is simply

$$D=\sum_{i=1}^{N} d_i \tag{8}$$

where

$$d_i=\frac{1}{2N_f(S_i)}=\frac{S_i^m}{2K}=\frac{2^m P_i^m}{2K} \tag{9}$$

is the damage caused by the $i$th half-cycle. In practice, failure is assumed when $D$ exceeds unity. The correspondence between stress ranges and peaks/valleys is valid only for a narrowband process. For wideband processes, it is necessary to apply a cycle counting algorithm such as rainflow counting [18] to distill the stress cycles from the time series. It is well known that Miner's rule is a convenient approximation and there is significant model uncertainty associated with it. One limitation is that it does not consider the sequence of the stress reversals. Wirsching and Chen [1] proposed that the uncertainty in the fatigue life prediction due to Miner's rule can be represented by a lognormal random variable with a CoV of 0.30.

The mean of $d_i$ is formulated as

$$\overline{d}_i=\frac{2^m}{2K}\int_0^\infty p^m f_P(p)\text{d}p. \tag{10}$$

By using Eq. (6), the integral can be evaluated to give [2]

$$\overline{d}_i=\frac{1}{2K}(2\sqrt{2}\sigma_X)^m\Gamma\left(1+\frac{m}{2}\right). \tag{11}$$

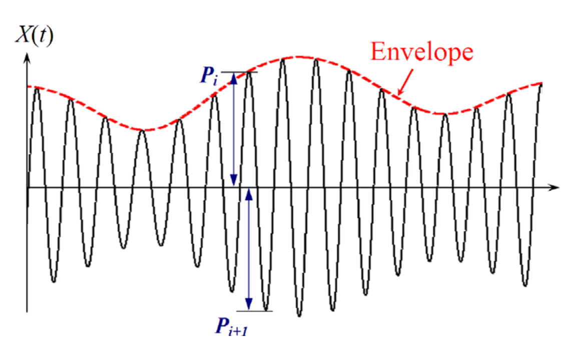


**Fig. 1.** Illustration of a narrowband process, its envelope and the damage components.