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Generation of non-Gaussian stochastic processes using nonlinear filters

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ABSTRACT

Non-Gaussian stochastic processes are generated using nonlinear filters in terms of Itô differential equations. In generating the stochastic processes, two most important characteristics, the spectral density and the probability density, are taken into consideration. The drift coefficients in the Itô differential equations can be adjusted to match the spectral density, while the diffusion coefficients are chosen according to the probability density. The method is capable to generate a stochastic process with a spectral density of one peak or multiple peaks. The locations of the peaks and the band widths can be tuned by adjusting model parameters. For a low-pass process with the spectrum peak at zero frequency, the nonlinear filter can match any probability distribution, defined either in an infinite interval, a semi-infinite interval, or a finite interval. For a process with a spectrum peak at a non-zero frequency or with multiple peaks, the nonlinear filter model also offers a variety of profiles for probability distributions. The non-Gaussian stochastic processes generated by the nonlinear filters can be used for analysis, as well as Monte Carlo simulation.

1. Introduction

Stochastic processes are involved in many areas, such as physics, engineering, ecology, biology, medicine, psychology, and other disciplines. For purposes of analysis and simulation, stochastic processes are required to be properly modeled and generated mathematically. It is important that a stochastic process to be generated should resemble its measured or estimated statistical and probabilistic properties. Two important measures have been used for the purpose: the probability distribution and the power spectral density. Conceptually, the probability distribution is the property at one time instant, and it is the first-order property of the process. The power spectral density, on the other hand, is a statistical property involving two different time instants, and hence is a second-order property. For a stationary stochastic process, both the probability distribution and the power spectral density are invariant with time.

To describe the probability distribution, many mathematical models have been proposed, defined either in infinite intervals, semi-infinite intervals, or finite intervals. Although a probability distribution defined in an infinite interval or a semi-infinite interval is not realistic since any real physical quantity is always bounded, it is still widely used due to its simplicity in mathematical treatment. However, when adoption of such distributions may affect the analysis significantly, models of probability distributions defined in finite ranges become necessary.

Another important property of a stochastic process, the power spectral density, describes the energy distribution of the process in frequency domain. It may be more important than the probability distribution in some situations [1]. For a Gaussian process, mathematical models can be obtained to match any given spectral density [2]. However, generation of a stochastic process with a non-Gaussian distribution and a given spectral density is much more complicated [3–5]. This is one of the reasons for the popularity of the Gaussian distribution, besides its simplicity in mathematical treatment. For many practical stochastic processes, the probability distributions are far from Gaussian, and adoption of Gaussian distribution may results in large errors.

Based on the above considerations, versatile models for a stochastic process are needed with the non-Gaussian probability distribution and spectral density available. The method of nonlinear filters was proposed for this purpose [6,7], in which the Itô type stochastic differential equations are employed with the drift coefficients adjusted to match the spectral density and the diffusion coefficients determined to fit the probability density. The present paper reviews the method systematically and extends the procedure for modeling stochastic processes with more profiles for the probability distributions. For overcoming the difficulty in the Monte Carlo simulation for certain types of bounded stochastic processes, equivalent models and alternative procedures are presented.

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2. Low-pass processes

Consider a stationary stochastic process $X(t)$ generated from the following Itô stochastic differential equation [8]:

$$dX = -\alpha X dt + D(X)\, dB(t) \tag{1}$$

where α is a positive constant, $B(t)$ is a unit Wiener process, and function $D(X)$ will be determined subsequently according to the properties of $X(t)$. Multiplying (1) by $X(t-\tau)$ and taking the ensemble average, we obtain

$$\frac{dR_{XX}(\tau)}{d\tau} = -\alpha\, R_{XX}(\tau) \tag{2}$$

where $R_{XX}(\tau) = E\,[X(t)X(t-\tau)]$ is the correlation function of $X(t)$. Let the mean-square value of $X(t)$ be

$$R_{XX}(0) = E[X^2(t)] = \sigma^2 \tag{3}$$

which is the initial condition for Eq. (2). The solution of Eq. (2) is given by

$$R_{XX}(\tau) = \sigma^2 exp\,(-\alpha|\tau|) \tag{4}$$

The corresponding spectral density of $X(t)$, i.e., the Fourier transform of $R_{XX}(\tau)$, is of the low-pass type

$$\Phi_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} d\tau = \frac{\alpha\,\sigma^2}{\pi(\omega^2+\alpha^2)} \tag{5}$$

Eq. (5) shows that the spectral density reaches its peak at zero, and the band width is controlled by parameterα.

The stationary probability density $p(x)$of $X(t)$ is governed by the reduced Fokker–Planck equation

$$\frac{d}{dx}G(x) = \frac{d}{dx}\left\{\alpha x p(x) + \frac{1}{2}\frac{d}{dx}[D^2(x)p(x)]\right\} = 0 \tag{6}$$

where $G(x)$ is known as the probability flow. In the present one-dimensional case, $G(x)$ must vanish everywhere [9], and Eq. (6) reduces to

$$\alpha x p(x) + \frac{1}{2}\frac{d}{dx}[D^2(x)p(x)] = 0 \tag{7}$$

Eq. (7) leads to [6]

$$D^2(x) = -\frac{2\alpha}{p(x)}\int x p(x) dx \tag{8}$$

or in the form of definite integration

$$D^2(x) = -\frac{2\alpha}{p(x)}\int_{x_l}^{x} u p(u) du \tag{9}$$

where x_l is the left boundary of the defining range of the stochastic process $X(t)$. It is shown [6] that the mean value of $X(t)$ must be zero to guarantee the left-hand side to be non-negative.

Thus, the stochastic process $X(t)$ generated from (1) with $D(x)$ given by (8) or (9) possesses a given stationary probability density $p(x)$ and a low-pass spectral density (5). The parameter α can be used to adjust the spectral density, and function $D(x)$ is used to match any valid probability distribution.

It is noted that the probability distribution is Gaussian if $D(x)$ is a constant; otherwise, the distribution is non-Gaussian. Several commonly used probability distributions will be discussed below.

2.1. Stochastic processes defined in an infinite interval

For easy mathematical treatment, stochastic processes are often assumed to be defined in an infinite interval $(-\infty,\infty)$. A versatile form of the probability distribution is of the following exponential form:

$$p(x) = C\exp[-\phi(x)] \tag{10}$$

where function $\phi(x)$ is known as the probability potential. It can be any function as long as it satisfies certain conditions so that (10) is a valid probability density function. Substituting (10) into (8), we obtain

$$D^2(x) = -2\alpha\exp[\phi(x)]\int x\exp[-\phi(x)]dx \tag{11}$$

as an example, consider

$$\phi(x) = \gamma x^2 + \delta x^4, \quad -\infty < x < \infty \tag{12}$$

where $\gamma > 0$ if $\delta = 0$, or γ is arbitrary if $\delta > 0$. Substitution of (12) into (11) results in

$$D^2(x) = \frac{\alpha\sqrt{\pi}}{2\sqrt{\delta}}\exp\left[\delta\left(x^2+\frac{\gamma}{2\delta}\right)^2\right] erfc\left[\delta\left(x^2+\frac{\gamma}{2\delta}\right)^2\right] \tag{13}$$

where erfc $(\cdot)$ is the complementary error function defined as

$$erfc(y) = \frac{2}{\sqrt{\pi}}\int_y^{\infty} e^{-t^2} dt \tag{14}$$

Another type of possible probability distribution is of the power form

$$p(x) = C(1+\gamma x^2)^{-\sigma}, \quad -\infty < x < \infty, \quad \gamma > 0 \tag{15}$$

application of (8) leads to

$$D^2(x) = \frac{\alpha}{\gamma(\sigma-1)}(1+\gamma x^2) \tag{16}$$

which requires that $\sigma > 1$. The Itô equation to generate stochastic process $X(t)$ is then

$$dX = -\alpha X dt + \sqrt{\frac{\alpha}{\gamma(\sigma-1)}(1+\gamma X^2)}\, dB(t) \tag{17}$$

2.2. Stochastic processes defined in a semi-infinite interval

Consider a stochastic process $X(t)$ defined in a semi-definite domain $[0,\infty]$. Its mean value is not zero; hence, there is a delta function at zero frequency in the power spectral density. In such a case, the above scheme of using nonlinear filters is not applicable directly. However, we may define a zero-mean process $Y(t)$ by shifting $X(t)$, i.e.,

$$Y = X - \mu_X, \quad -\mu_X \le Y < \infty \tag{18}$$

where μ_X is the mean value of $X(t)$. Thus, a nonlinear filter can be constructed to generate process $Y(t)$, and then $X(t)$ from (18).

An example is the exponential distribution expressed as

$$p(x) = ae^{-ax}, \quad a > 0, \quad 0 \le x < \infty \tag{19}$$

with a mean value of $1/a$. The probability density for $Y(t)$ is

$$p(y) = ae^{-(ay+1)}, \quad -\frac{1}{a} \le y < \infty \tag{20}$$

function $D^2(y)$ are then obtained from Eq. (8) as

$$D^2(y) = \frac{2\alpha}{a}\left(y+\frac{1}{a}\right) \tag{21}$$

the Itô equation to generate $Y(t)$ and $X(t)$ are respectively,

$$dY = -\alpha\, Y dt + \sqrt{\frac{2\alpha}{a}\left(Y+\frac{1}{a}\right)}\, dB(t) \tag{22}$$

$$dX = -\alpha\left(X-\frac{1}{a}\right)dt + \sqrt{\frac{2\alpha}{a}X}\, dB(t) \tag{23}$$

note that the spectral density of $X(t)$ contains a delta function due to its nonzero mean.

Another example is the Rayleigh distribution given by

$$p(x) = \gamma^2 x\exp(-\gamma x), \quad 0 \le x < \infty \tag{24}$$